Heisenberg Equations for Non-Degenerate Two-Photon Jaynes-Cummings Model with Dynamic Stark-Shifts

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The problem of the non-degenerate two-photon Jaynes-Cummings model, with the Stark-shifts is investigated. The Heisenberg equations for the relevant operators are derived by using constant of motion method. Especially, a derivation of the operator differential equation for a product $a_{12}(t) = a_1(t)a_2(t)$ will be shown in detail.

I. Introduction

One of the simplest models for the interactions between an atom and a quantized electromagnetic field is known as the Jaynes-Cummings model (JCM)\(^1\).

In this model, the atom is assumed to have two levels, and the electromagnetic field is a single mode. For the past several years, we have investigated the model for the single-photon, the two-photon, and the multi-photon cases. First, we studied the characteristic functions and their time behaviors for a simple JCM\(^2\) using the density matrix approach, and then, by using the Heisenberg equation method, we examined the degenerate two-photon JCM\(^3\), where the operator differential equations and their solutions are obtained. We further studied the cases of the non-degenerate two-photon JCM\(^4\) and the multi-photon JCM\(^5\). Moreover, the atomic spectrum for multi-photon JCM\(^6\) and the field spectrum for multi-photon JCM\(^7\) were explored, and these spectra were presented in the 3-dimensional pictures. We also solved the degenerate two-photon JCM with Stark-shifts\(^8\), and further we just published the results of the non-degenerate two-photon JCM with Stark-shifts\(^9\).

In this paper, we show a derivation of the product operator $a_{12}(t) = a_1(t)a_2(t)$. We explicitly use the idea of the constant motion operator. Since the derivations are quite complicated and it is not straightforward, we showed them in detail here.

The present paper is organized as follows. In section II, we show the effective non-degenerate Hamiltonians and the constant of motion operators. In section III, the second-order differential equation for a product operator $a_{12}(t) = a_1(t)a_2(t)$ of Mode 1 and Mode 2, will

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be derived, and some detail derivations are presented. In section IV, the differential equations for the atomic and field operators are presented, and their derivations are shown in APPENDIX. The conclusions and discussions are prepared in section V.

II. Effective Hamiltonians and Constants of Motions

An interaction between a two-mode field and an effective two-level atom in the cavity is considered, where we assumed that the two-photon transition between the atom and the field is allowed. Note that the upper and lower atomic states are denoted by \( |b\rangle \) and \( |a\rangle \), respectively. We also assume an intermediate virtual state, indicated as \( |i\rangle \), which is governed solely by the cavity dimension.

The effective Hamiltonian of the system in the rotating-wave approximation (RWA) is written by \( (\hbar = 1) \)

\[
H = H_A + H_F + H_S + H_I.
\]  
(1)

where \( H_A, H_F, H_S, \) and \( H_I \) are the atomic, the field, the Stark-shifts, and the interaction Hamiltonians, respectively. The atomic Hamiltonian is expressed as

\[
H_A = \frac{1}{2} \omega_0 \sigma_z,
\]

where \( \omega_0 \) is the energy difference of the two atomic levels with choosing the zero energy level for the midway between them, and \( \sigma_z \) is a z-component of the atomic Pauli spin operator. On the other hand, for the Hamiltonians of the field, of the Stark-shifts, and of the interaction, we have, respectively,

\[
H_F = \omega_1 n_1 + \omega_2 n_2.
\]

(3)

\[
H_S = \frac{1}{2} \beta_a (1 - \sigma_z) n_1 + \frac{1}{2} \beta_b (1 + \sigma_z) n_2,
\]

(4)

and

\[
H_I = g (a_1 a_2 \sigma^+ + a_1^* a_2^* \sigma^-),
\]

(5)

where the subscripts 1 and 2 on the number and photon operators, and on the frequencies are associated with the cavity field Mode 1 and Mode 2, respectively. Further, letters \( \sigma^+ \) and \( \sigma^- \) are the raising and lowering atomic level operators. Note also that letters \( \beta_a \) and \( \beta_b \) are Stark-shifts parameters associated with lower and upper atomic levels, respectively.

In order to formulate the Heisenberg equations of motion of the respective operators, we will rewrite the total Hamiltonian for a convenient form, such as

\[
H = \omega_1 N_1 + \omega_2 N_2 + D.
\]

(6)

where \( N_1, N_2, \) and \( D \) are so-called the constant of motion operators, and can be expressed as

\[
N_1 = n_1 + \frac{1}{2} \sigma_z,
\]

\[
N_2 = n_2 + \frac{1}{2} \sigma_z.
\]

(7)
and
\[ D = \frac{1}{2} \beta_1 (1 - \sigma_z) n_1 + \frac{1}{2} \beta_2 (1 + \sigma_z) n_2 + g (a_1 a_2 \sigma^+ + a_1^\dagger a_2^\dagger \sigma^-). \]  
\(8\)

Note that the method which we will use in this paper is the one introduced by Arkerhalt and Rzażewski\(^{10}\). It can easily be shown that the operators \(N_1\), \(N_2\), and \(D\) commute each other and with Hamiltonian, so that
\[
[N_1, N_2] = [N_1, D] = [N_2, D] = [H, N_1] = [H, N_2] = [H, D] = 0. \tag{9}
\]

They are conserved quantities and are constants with respect to time, and play a crucial role in constructing the operator differential equations, which we will show in following sections.

III. Heisenberg Equations

The Heisenberg equations is known as
\[
\dot{O}(t) = -i \left[ O(t), H \right], \tag{10}
\]
where \(O(t)\) is an arbitrary operator.

We now construct the differential equation for the simultaneous photon annihilation operator, which we denote
\[
a_{12}(t) = a_{1}(t)a_{2}(t). \tag{11}
\]

Since the interchanges of the order of operators are strictly prohibited, the formulation of the second-order differential equation of the product operator \(a_{12}(t)\) will be quite complicated. It must be noted that the derivation of the equation \(\dot{a}_{12}(t)\) requires some other first order differential equations. First, we will present the first-order equations for the photon annihilation operators of the modes as
\[
\dot{a}_1 = -i \left[ a_1, H \right] = -i \left[ a_1, \omega_1 N_1 + \omega_2 N_2 + D \right] = -i \left( \omega_1 + \frac{1}{2} \beta_1 (1 - \sigma_z) \right) a_1 - i g a_1^\dagger \sigma^-, \tag{12}
\]
for the photon annihilation operator of Mode 1, and similarly
\[
\dot{a}_2 = -i \left( \omega_2 + \frac{1}{2} \beta_2 (1 + \sigma_z) \right) a_2 - i g a_2^\dagger \sigma^- , \tag{13}
\]
for Mode 2. On the other hand, the first order differential equation of the atomic dipole-moment operator, denoted as \(\sigma^-(t)\), is
\[
\dot{\sigma}^- = -i \left[ \sigma^-, H \right] = -i \left[ \sigma^-, \omega_1 N_1 + \omega_2 N_2 + D \right] = -i (\omega_1 + \omega_2) \sigma^- - i \left[ \sigma^-, D \right] = -i (\omega_1 + \omega_2 - 2D) \sigma^- - i (D \sigma^- + \sigma^- D) \tag{14}
\]
For the last term of the above equation, we have
\[
D\sigma^- + \sigma^- D = \beta_\sigma \sigma^- n_1 + ga_1 a_2 \sigma^+ \sigma^- + \beta_\sigma \sigma^- n_2 + ga_1 a_3 \sigma^- \sigma^+ \\
= (\beta_\sigma \sigma^- n_1 + \beta_\sigma n_2) \sigma^- + ga_1 a_2 (\sigma^+ \sigma^- + \sigma^- \sigma^+) \\
= (\beta_\sigma N_1 + \beta_\sigma N_2 + \beta^+) \sigma^- + ga_1 a_2
\]

where we have used \( \sigma^+ \sigma^- + \sigma^- \sigma^+ = 1 \), and a new symbol \( \beta^+ = \frac{\beta_\sigma + \beta_\delta}{2} \) was substituted. Thus we finally obtain from Eq. (14)

\[
\dot{\sigma}^- = -i (\omega_1 + \omega_2 - 2D + \beta_\sigma N_1 + \beta_\sigma N_2) \sigma^- - ig a_1 a_2
\]

For a later use, we have to present an alternative form of \( \sigma^- \), i.e.,

\[
\dot{\sigma}^- = -i (\omega_1 + \omega_2) \sigma^- - i [\sigma^-, D].
\]

The second term can be calculated, and we have

\[
\dot{\sigma}^- = -i (\omega_1 + \omega_2 + \beta_\sigma N_2 - \beta_\sigma N_1 + \beta^-) \sigma^- + ig a_1 a_2 \sigma_z,
\]

where we have \( \beta^- = \frac{\beta_\sigma - \beta_\delta}{2} \). The z-component of the atomic operator is also needed in order to establish the equation of \( \dot{a}_{12}(t) \), so we calculate

\[
\dot{\sigma}_z = -i [\sigma_z, D] = 2i D \sigma_z - i (\sigma_z D + D \sigma_z),
\]

the derivation of the second term of above is presented in APPENDIX A. Thus we found

\[
\dot{\sigma}_z = i (2D + \beta^- - (\beta_\sigma N_1 + \beta_\sigma N_2)) \sigma_z + i (\beta_\sigma N_1 - \beta_\sigma N_2 + \beta^+),
\]

Up to here, we obtained the first-order differential equations for respective operators.

We now construct the equation for \( \dot{a}_{12}(t) \). Though the derivation of the equation is quite complicated, the formation of the second-order operator differential equation of \( a_{12}(t) \) is a main theme of this paper. Thus we present the derivations here in detail. First, by using Heisenberg equation we have

\[
\dot{a}_{12} = -i [a_{12}, H] = -i (\omega_1 + \omega_2) a_{12} - i [a_{12}, D].
\]

The commutator in the above equation is calculated as

\[
[a_{12}, D] = [a_{12}, \frac{\beta_\sigma}{2} (1 - \sigma_z) n_1 + \frac{\beta_\delta}{2} (1 + \sigma_z) n_2 + g (a_1^+ a_2 \sigma^+ + a_1 a_2^+ \sigma^-) ]
\]

\[
= \frac{\beta_\sigma}{2} (1 - \sigma_z) [a_1 a_2 n_1] + \frac{\beta_\delta}{2} (1 + \sigma_z) [a_1 a_2 n_2] + g [a_1 a_2, a_1^+ a_2^+ \sigma^-]
\]

\[
= \frac{\beta_\sigma}{2} (1 - \sigma_z) a_1 a_2 + \frac{\beta_\delta}{2} (1 + \sigma_z) a_1 a_2 + g (n_1 + n_2 + 1) \sigma^-
\]

\[
= (\beta^+ + \beta^- \sigma_z) a_1 a_2 + g (N_1 + N_2 + 2) \sigma^-,
\]

thus we have

\[
\dot{a}_{12} = -i (\omega_1 + \omega_2 + \beta^+ + \beta^- \sigma_z) a_{12} - ig (N_1 + N_2 + 2) \sigma^-.
\]
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Note that Eq. (23) contains not only constants of motion but also bare operators, namely $\sigma_z$ and $\sigma^-$, so that we need the second-order differential equation. In order to establish the second-order equation, we arrange the above equation as

$$\left(\frac{d}{dt} + i (\omega_1 + \omega_2 + \beta^+ + \beta^- \sigma_z)\right) a_{12} = -i g (N_1 + N_2 + 2) \sigma^-.$$  \hfill (24)

Note that from Eq. (26), we also have

$$\left(\frac{d}{dt} + i (\omega_1 + \omega_2 - 2D + \gamma)\right) \sigma^- = -iga_{12},$$  \hfill (25)

where $\gamma = \beta_2 N_1 + \beta_3 N_2 + \beta^+$. We now construct the equation

$$\left(\frac{d}{dt} + i (\omega_1 + \omega_2 - 2D + \gamma)\right) \left(\frac{d}{dt} + i (\omega_1 + \omega_2 + \beta^+ + \beta^- \sigma_z)\right) a_{12}$$

$$= -g^2(N_1 + N_2 + 2) a_{12}. \hfill (26)$$

If the term containing $\sigma_z$ is extracted from Eq. (26), we have

$$a_{12} + i (2 (\omega_1 + \omega_2 - D) + \gamma + \beta^+) \dot{a}_{12}$$

$$- ((\omega_1 + \omega_2 + \beta^+)(\omega_1 + \omega_2 - 2D + \gamma) + g^2(N_1 + N_2 + 2)) a_{12}$$

$$+ i\beta^- \sigma_z a_{12} + i\beta^- \dot{\sigma}_z a_{12} \hfill A$$

$$+ -\beta^- (\omega_1 + \omega_2 - 2D + B) \sigma_z a_{12} = 0. \hfill (27)$$

We first calculate the term denoted by $C$

$$C = -\beta^- (\omega_1 + \omega_2 + \beta^+ + \beta^- - 2D - \beta^- + (\beta_2 N_1 + \beta_3 N_2)) \sigma_z a_{12}$$

$$= -\beta^- (\omega_1 + \omega_2 + \beta^+ + \beta^-) \sigma_z a_{12} \hfill E$$

$$+ \beta^- (2D + \beta^- - (\beta_2 N_1 + \beta_3 N_2)) \sigma_z a_{12}, \hfill (28)$$

then we have

$$A + E = i\beta^- \sigma_z (a_{12} + i (\omega_1 + \omega_2 + \beta^+) a_{12} + i\beta^- a_{12})$$

$$= i\beta^- \sigma_z (-i\beta^- \sigma_z a_{12} - ig(N_1 + N_2 + 2) \sigma^- + i\beta^- a_{12})$$

$$= i\beta^- (-\dot{a}_{12} - i (\omega_1 + \omega_2 + \beta^+) a_{12} - i\beta^- a_{12})$$

$$= -i\beta^- \dot{a}_{12} + \beta^- (\omega_1 + \omega_2 + \beta^+ + \beta^+) a_{12}, \hfill (29)$$

where Eq. (23) was used. Further, we calculate

$$B + F = i\beta^- \dot{\sigma}_z - i \left[2D + \beta^- (\beta_2 N_1 + \beta_3 N_2)\right] \sigma_z a_{12}$$

$$= -\beta^- (\beta_2 N_1 - \beta_3 N_2 + \beta^+) a_{12} \hfill (30)$$

where we have used Eq. (23). If we combine above equations, we have

$$A + E + B + F = -i\beta^- \dot{a}_{12} + \beta^- (\omega_1 + \omega_2 + \beta_2 N_2 - \beta_3 N_1 + \beta^+) a_{12}. \hfill (31)$$
By substituting Eq. (31) into Eq. (27), we finally obtained the second-order equation for the operator $a_{12}(t)$ as

$$\dot{a}_{12} + 2i(\omega_1 + \omega_2 - \tau) \dot{a}_{12} + \chi a_{12} = 0,$$

where letters $\tau$ and $\chi$ are, respectively

$$\tau = 2D - \frac{1}{2}(\beta_a N_1 + \beta_a N_2) - \beta^+ + \frac{1}{2}\beta^-,$$

and

$$\chi = g^2(N_1 + N_2 + 2) + \beta^- (\omega_1 + \omega_2 + \beta_a N_2 - \beta_a N_1 + \beta^-) - (\omega_1 + \omega_2 + \beta^+)(\omega_1 + \omega_2 - 2D + \beta_a N_2 + \beta_a N_1 + \beta^+).$$

Note that $\tau$ and $\chi$ contain simple parameters and some constants of motions. Therefore, the second order differential equation of $a_{12}(t)$ can be solved.

**IV. Differential Equations for Other Operators**

In this section, we merely write down the second-order operator differential equations for respective operators, such as $a_1(t)$, $a_2(t)$, and $\sigma^-(t)$, and detail derivations are presented in APPENDICES.

The second-order differential equation of $a_1(t)$ is (See APPENDIX B)

$$\dot{a}_1 + 2i(\omega_1 - \zeta_1) \dot{a}_1 + \eta_1 a_1 = 0,$$

similarly, for $a_2(t)$

$$\dot{a}_2 + 2i(\omega_2 - \zeta_2) \dot{a}_2 + \eta_2 a_2 = 0,$$

where

$$\zeta_1 = D - \frac{1}{2}(\beta_a N_2 + \beta_a N_1) - \frac{1}{2}(\beta_a - \beta^-),$$

$$\zeta_2 = D - \frac{1}{2}(\beta_a N_2 + \beta_a N_1) - \frac{1}{2}(\beta_a - \beta^-),$$

$$\eta_1 = g^2(N_2 + \frac{1}{2}) - \omega_1(\omega_1 - 2D + \beta_a N_2 + \beta_a N_1 + \beta_a - \beta^-) + \frac{1}{2} \beta_a(2D - 2\beta_a N_2 + \beta_a),$$

and

$$\eta_2 = g^2(N_2 + \frac{1}{2}) - \omega_2(\omega_2 - 2D + \beta_a N_2 + \beta_a N_1 + \beta_a - \beta^-) + \frac{1}{2} \beta_a(2D - 2\beta_a N_2 + \beta_a).$$

For $\sigma^-(t)$, we obtained (See APPENDIX C)
\[ \hat{\sigma}^- + 2i(\omega_1 + \omega_2 - \mu) \hat{\sigma}^- + \nu \hat{\sigma}^+ = 0, \]  

where the coefficient operators \( \mu \) and \( \nu \) are represented as

\[ \mu = D - 1/2(\beta_+ N_1 + \beta_- N_2) - \beta^+ - 1/2 \beta^- , \]

and

\[ \nu = g^2(N_1 + N_2 + 2) - (\omega_1 + \omega_2 + \beta^+) (\omega_1 + \omega_2 - 2D + \beta_+ N_1 + \beta_- N_2 + \beta^+) \]

\[ - \beta^- (\omega_1 + \omega_2 + \beta_+ N_2 - \beta_- N_1 + \beta^-) . \]

**V. Conclusions**

In this paper, the problem of the non-degenerate two-photon Jaynes-Cummings model with the Stark-shifts is investigated. We have formulated the second-order operator differential equation of simultaneous photon annihilation operator \( a_1(t) a_2(t) \). Even if there are several kinds of two-photon JCM, we found that the problem of the non-degenerate two-photon JCM with Stark-shifts is the most complicated one. We had to deal with four dynamic operators, such as \( a_1(t) \), \( a_2(t) \), \( \sigma^-(t) \), and \( \sigma_z(t) \). However we successfully could formulate the differential equations by using the so-called constants of motion operator method. Though the solution of the equation is not yet found, it is under investigation now. We have just solved the problem of non-degenerate two-photon JCM with dynamic Stark-shifts included, and published the results elsewhere\(^9\). Since the detail derivations of the equations are not shown in Ref\(^9\), we left that in APPENDICES of this paper.

It should be noted that, for the case of the absence of the Stark-shifts, all equations obtained in this paper become those in Ref\(^{41}\).

**Appendix A**

**Derivation of \( \sigma_z D \) and \( D \sigma_z \)**

We first calculate

\[ \sigma_z D = \sigma_z \left( \frac{\beta_+}{2} (1 + \sigma_z) n_2 + \frac{\beta_-}{2} (1 - \sigma_z) n_1 + g (a_1 a_2 \sigma^+ + a_1^* a_2^* \sigma^-) \right) \]

\[ = \frac{\beta_+}{2} (\sigma_z + 1) n_2 + \frac{\beta_-}{2} (\sigma_z - 1) n_1 + g (a_1 a_2 \sigma_z \sigma^+ + a_1^* a_2^* \sigma_z \sigma^-), \]  

(A.1)

and \( D \sigma_z \) is shown as

\[ D \sigma_z = (1 + \sigma_z) n_2 + \frac{\beta_+}{2} (1 - \sigma_z) n_1 + g (a_1 a_2 \sigma^+ + a_1^* a_2^* \sigma^-) \sigma_z \]

\[ = \frac{\beta_+}{2} (\sigma_z + 1) n_2 + \frac{\beta_-}{2} (\sigma_z - 1) n_1 + g (a_1 a_2 \sigma_z \sigma^+ + a_1^* a_2^* \sigma_z \sigma^-). \]  

(A.2)

If we combine above two equations, we obtain

\[ \sigma_z D + D \sigma_z = (\beta_+ N_2 + \beta_- N_1 - \beta^-) \sigma_z + (\beta_+ N_2 - \beta_- N_1 - \beta^+) \]

(A.3)

Thus we have Eq. \( \Box \).
Appendix B
Derivation of $\dot{a}_l(t)$

By taking the derivative of Eq.(12), we have

$$\dot{a}_l = -iV\dot{a}_l + i\frac{\beta_s}{2} \sigma_z \sigma_l - i\alpha a^\dagger_l \sigma^+ - i\beta a^\dagger_l \sigma^-.$$  \hfill (B.1)

We first calculate

$$G = -i\alpha a^\dagger_l \sigma^- = \left(-i \left[ \omega_2 + \frac{\beta_s}{2} (1 + \sigma_z) \right] a^\dagger_l \alpha a^\dagger_l \sigma^+ \right) \sigma^-$$

$$= i \left( \omega_2 + \frac{\beta_s}{2} (1 + \sigma_z) \right) a_1 - \left( \omega_2 + \frac{\beta_s}{2} (1 + \sigma_z) \right) \left( \omega_1 + \frac{\beta_s}{2} (1 - \sigma_z) \right) a_1$$

$$+ g^2 a_1 \sigma^+ \sigma^-,$$  \hfill (B.2)

where we have used Eq. (13). By using Eqs. (12) and (16), $J$ will be

$$J = -i \alpha a^\dagger \sigma^-$$

$$= -ga^\dagger \left( \omega_1 + \omega_2 - 2D + \beta_s N_1 + \beta_s N_2 + \beta^+ \right) \sigma^- - g^2 n_2 a_1$$

$$= -ga^\dagger \left( \omega_1 + \omega_2 + \beta_s N_1 + \beta^+ \right) \sigma^- - ga^\dagger (-2D + \beta_s N_2) \sigma^- - g^2 n_2 a_1$$

$$= -i \left( \omega_1 + \omega_2 + \beta_s N_1 + \beta^+ \right) (ga^\dagger \sigma^- - ig(2a^\dagger D - i\beta_a a^\dagger N_2) \sigma^- - g^2 n_2 a_1$$

$$= -i \left( \omega_1 + \omega_2 + \beta_s N_1 + \beta^+ \right) \left( a_1 + i \left( \omega_1 + \frac{\beta_s}{2} (1 - \sigma_z) \right) a_1 \right)$$

$$- ig^2 n_2 a_1 - ig \left( 2i a^\dagger D - i\beta_a a^\dagger N_2 \right) \sigma^-.$$  \hfill (B.3)

Now we calculate the term $a^\dagger \sigma^+ D$ in $K$,

$$a^\dagger \sigma^+ D = D a^\dagger + [a^\dagger, D] = D a^\dagger + [a^\dagger, \frac{1}{2} \beta_s (1 + \sigma_z) n_2 + ga_a a^\dagger \sigma^+]$$

$$= D a^\dagger - \frac{\beta_s}{2} (1 + \sigma_z) a^\dagger - ga_a \sigma^+,$$  \hfill (B.4)

and similarly,

$$\beta a^\dagger \sigma^+ \sigma^- = \beta a^\dagger \left( n_2 + \frac{\sigma_z}{2} \right) a^\dagger = \beta \left( n_2 - 1 \right) + \frac{\sigma_z}{2} a^\dagger = \beta \left( N_2 - 1 \right) a^\dagger.$$  \hfill (B.5)

By combining the above equations, we have for $K$

$$K = -ig \left( 2i a^\dagger D - i\beta_a a^\dagger N_2 \right) \sigma^-$$

$$= -ig \left[ 2i \left( Da^\dagger - \frac{\beta_s}{2} (1 + \sigma_z) a - ga_a \sigma^+ \right) - i\beta \left( N_2 - 1 \right) a^\dagger \right] \sigma^-$$

$$= ig \left[ 2D - \beta_s (1 + \sigma_z) - \beta \left( N_2 - 1 \right) \right] (-i\alpha a^\dagger \sigma^-) - ig (-2iga_a \sigma^+ \sigma^-)$$

$$= i \left( 2D - \beta_s N_2 - \beta \sigma_z \right) a_1 - \left( 2D - \beta_s N_2 - \beta \sigma_z \right) \left( \omega_1 + \frac{\beta_s}{2} (1 - \sigma_z) \right) a_1$$

$$- 2g^2 a_1 \sigma^+ \sigma^-.$$  \hfill (B.6)
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Thus $J$ will be

$$ J = -i(\omega_1 + \omega_2 - 2D + \beta_eN_1 + \beta_bN_2 + \beta^+ + \beta_0\sigma_z)\hat{a}_1 $$

$$ + (\omega_1 + \omega_2 - 2D + \beta_eN_1 + \beta_bN_2 + \beta^+ + \beta_0\sigma_z)(\omega_1 + \frac{\beta a}{2}(1 - \sigma_z))\hat{a}_1 $$

$$ - g^2n_0\hat{a}_1 - 2g^2a_1\sigma^+\sigma^- $$

(B.7)

By inserting Eqs. (B.2) and (B.7) into eq. (B.1), we obtain

$$ \hat{a}_1 = -i(2\omega_1 - 2D + \beta_eN_1 + \beta_bN_2 + \beta_0\sigma_z)\hat{a}_1 - g^2(N_2 + \frac{1}{2})\hat{a}_1 $$

$$ + \left(\frac{\beta_a}{4}(\omega_1 - 2D + \beta_eN_1 + \beta_bN_2 + \beta_0\sigma_z)\frac{-\beta_0}{2}\sigma_z\hat{a}_1 \right) $$

$$ + \frac{\beta_0}{2}(\omega_1 + \frac{\beta a}{2})\sigma_z\hat{a}_1 - \left(\omega_1 2D + \beta_eN_1 + \beta_bN_2 + \frac{\beta a}{2}\right)\frac{\beta a}{2}\sigma_z\hat{a}_1 $$

$$ + \frac{i\beta a}{2}\sigma_z. $$

(B.8)

First we calculate the term $L$

$$ L = -i\beta^+\sigma_z\hat{a}_1 = -i\beta^+\sigma_z \left[ -i\left(\omega_1 + \frac{\beta a}{2}(1 - \sigma_z)\right)\hat{a}_1 - ig^2\sigma^- \right] $$

$$ = -\beta^-\omega_1\sigma_z\hat{a}_1 - \beta^-\frac{\beta a}{2}\sigma_z\hat{a}_1 + \beta^-\frac{\beta a}{2}\hat{a}_1 + i\beta^-(iga_1\sigma_z\sigma^-) $$

$$ = -\beta^-\omega_1\sigma_z\hat{a}_1 - \beta^-\frac{\beta a}{2}\sigma_z\hat{a}_1 + \beta^-\frac{\beta a}{2}\hat{a}_1 + i\beta^-(\hat{a}_1 + i\left(\omega_1 + \frac{\beta a}{2}(1 - \sigma_z)\hat{a}_1\right)) $$

$$ = -\beta^-\omega_1\sigma_z\hat{a}_1 - \beta^-\frac{\beta a}{2}\sigma_z\hat{a}_1 + \beta^-\frac{\beta a}{2}\hat{a}_1 + i\omega_1\beta^-\hat{a}_1 + i\beta^-\hat{a}_1 - \beta^-\frac{\beta a}{2}\hat{a}_1 + \beta^-\frac{\beta a}{2}\sigma_z\hat{a}_1 $$

$$ = -\beta^-\omega_1\sigma_z\hat{a}_1 - \beta^-\omega_1\hat{a}_1 + i\beta^-\hat{a}_1, $$

(B.9)

so that

$$ L + M = -\beta^-\omega_1\sigma_z\hat{a}_1 - \beta^-\omega_1\hat{a}_1 + i\beta^-\hat{a}_1 $$

$$ + \frac{\beta a}{2}(\omega_1 + \frac{\beta a}{2})\sigma_z\hat{a}_1 - \left(\omega_1 - 2D + \beta_eN_1 + \beta_bN_2 + \frac{\beta a}{2}\right)\frac{\beta a}{2}\sigma_z\hat{a}_1 $$

$$ = i\beta^-\hat{a}_1 - \beta^-\omega_1\hat{a}_1 + \frac{\beta a}{2}\beta^-\sigma_z\hat{a}_1 $$

$$ - (2D - \beta_eN_1 - \beta_bN_2)\frac{\beta a}{2}\sigma_z\hat{a}_1. $$

(B.10)
On the other hand, for \( P \) we have

\[
P = i \frac{\beta_s}{2} \left[ i \left( 2D + \beta^- - (\beta a N_1 + \beta_b N_2) \right) \sigma_z + i \left( \beta a N_1 - \beta_b N_2 + \beta^+ \right) \right] a_1
\]

\[
= - \frac{\beta_s}{2} (2D + \beta^- - \beta a N_1 + \beta_b N_2) \sigma_z a_1
\]

\[
- \frac{\beta_s}{2} (\beta a N_1 - \beta_b N_2 + \beta^+) a_1.
\]

(B.11)

By adding \( L, M, \) and \( P \), we have

\[
L + M + P = \frac{\beta_s}{2} (\beta a N_2 - \beta a N_1, \beta^+) a_1 + i \beta^- a_1 - \beta^- \omega_1 a_1
\]

(B.12)

The insertion of Eq. (B.13) into Eq. (B.8) yields

\[
\dot{a} = -i \left[ 2\omega_1 - 2D + \beta a N_1 + \beta_b N_2 + \beta^- \right] \dot{a}_1 - g^2 \left( N_2 + \frac{1}{2} \right) a_1
\]

\[
+ \left( \omega_1 - 2D + \beta a N_1 + \beta_b N_2 + \frac{\beta_s}{2} \right) \left( \omega_1 + \frac{\beta_s}{2} \right) - \frac{\beta_s \beta_s}{4} \right] a_1
\]

\[
+ \frac{\beta_s}{2} (\beta a N_2 - \beta a N_1 - \beta^-) a_1 + \beta^- \omega_1 a_1.
\]

(B.13)

Thus we obtain Eq. \( \Omega \).

Appendix C
Derivation of \( \sigma^- (t) \)

By taking the derivative of Eq. (16), we have

\[
\dot{\sigma}^- = -i \left( \omega_1 + \omega_2 - 2D + \beta a N_1 + \beta_b N_2 + \beta^- \right) \dot{a}_1 - g^2 (N_1 + N_2) \sigma^- + i g (\dot{a}_1 a_2 + a_1 \dot{a}_2).
\]

(C.1)

Then the term \( Q \) is

\[
Q = -ig \left(-i \left( \omega_1 + \omega_2 \right) - i \frac{\beta_s}{2} (1 - \sigma_z) - i \frac{\beta_s}{2} (1 + \sigma_z) \right) a_1 a_2 - g^2 (N_1 + N_2) \sigma^-
\]

\[
= -i \left( \omega_1 + \omega_2 + \beta^+ \right) (-iga_1 a_2) - i\beta^- (-iga_1 a_2) - g^2 (N_1 + N_2 + 2) \sigma^-
\]

\[
= -i \left( \omega_1 + \omega_2 + \beta^+ \right) (\sigma^- + i \left( \omega_1 + \omega_2 - 2D + \beta a N_1 + \beta_b N_2 \right) \sigma^-)
\]

+ i\beta^- (\sigma^+ + i \left( \omega_1 + \omega_2 + \beta_b N_2 - \beta a N_1 + \beta^- \right))

\[
- g^2 (N_1 + N_2 + 2) \sigma^-
\]

(C.2)

where we have used Eqs. \( \Omega \) and \( \Omega \).
Heisenberg Equations for Non-Degenerate Two-Photon Jaynes-Cummings Model with Dynamic Stark-Shifts

References