THE FD (FINITE DIFFERENCE) & LP (LINEAR PROGRAMMING) METHOD

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Abstract

By combining finite difference method with linear programming, a new optimization technique (Finite Difference & Linear Programming Method, or, the FD & LP Method) has been developed in order to control systems of differential equations with both equality or inequality constraints and an objective function. The generalization of the FD & LP method is presented in this research. The tractability in the initial and boundary conditions and the equality or inequality constraints makes sure that the method becomes an useful technique for several new types of boundary value problems.

1. Introduction

Finite difference & linear programming method (the FD & LP method) (1, 2, 3, 5) has been developed in order to control systems of partial differential equations with both equality or inequality constraints and an objective function. Such systems are frequently encountered in various engineering and scientific problems of control and optimal design and, especially, are of interest in control of field problems (heat conduction, diffusion-convection, electric or magnetic potential, etc.). In the development of the FD & LP method, the combined use of finite difference method and linear programming has been adopted. The finite difference method is one of the powerful numerical method for the solution of differential equations. Linear programming is one of the most frequently used mathematical methods of operations research. In the development of the FD & LP method the concepts of the decision variable and the state variable are adopted as in Bellman's dynamic programming and Pontryagin's maximum principle. The FD & LP method utilizes the advantages of the numerical techniques of both finite difference method and linear programming. Aguado and Remson[1] made the combined use of finite difference method with linear programming in the field of ground water management. The article is an excellent pioneering research associated with the FD & LP method. In this research the generalization of the FD & LP method and the application of the method to field problems are presented. In a manner similar to the FD & LP method, the FE (Finite Element) & LP (Linear Programming) method has been developed by the combined use of finite element method with linear programming. Futagami made several researches associated with the FE & LP method[2),3),4),5),6),7)].

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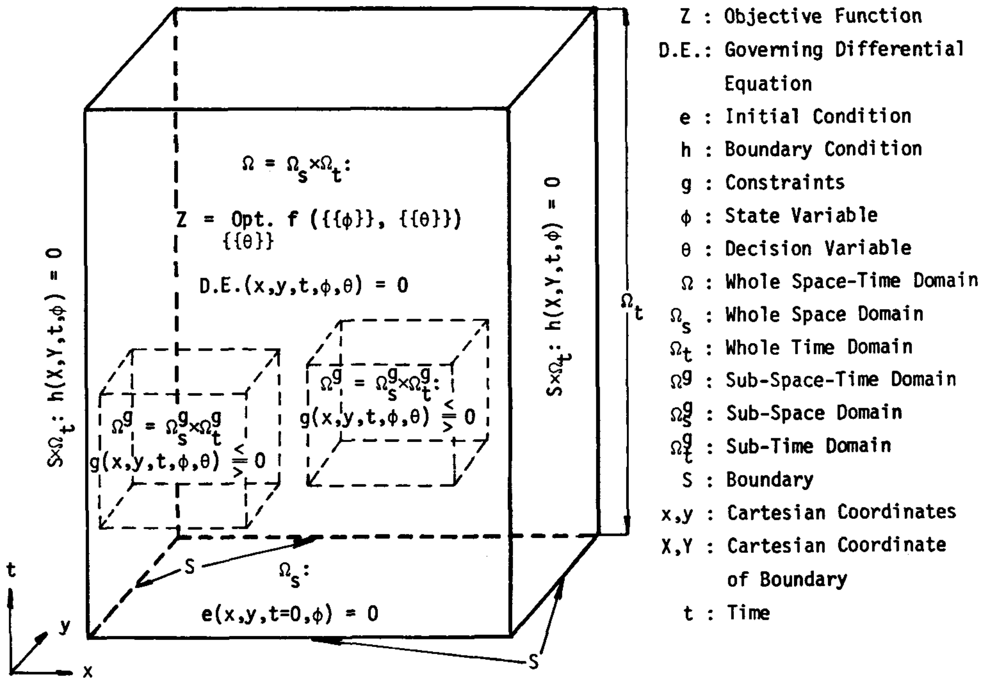

Fig. 1 General Concepts of the Finite Difference & Linear Programming Method

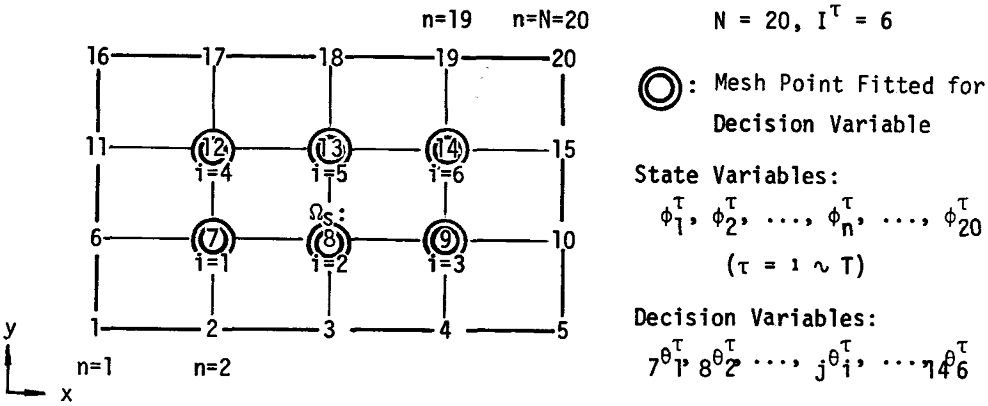

Fig. 2 Whole Space Domain Ω_s Divided into Finite Mesh Points

2. Systems of Basic Differential Equations

The FD (finite difference) & LP (linear programming) method has been developed and systematized in order to control the following systems of differential equations.

Objective Function (throughout the whole domain $(\Omega=\Omega_s\times\Omega_t)$*)*

$$Z=\underset{\{\{\theta\}\}}{Opt.}\, f(\{\{\phi\}\},\{\{\theta\}\})=\begin{cases}\underset{\{\{\theta\}\}}{Max.}\, f(\{\{\phi\}\},\{\{\theta\}\})\\ \underset{\{\{\theta\}\}}{Min.}\, f(\{\{\phi\}\},\{\{\theta\}\})\end{cases} \tag{1}$$

subject to:

Equilibrium Equations

Governing Differential Equation (in the whole domain $(\Omega=\Omega_s\times\Omega_t)$)

$$D.E.(\boldsymbol{x},\ t,\ \phi,\ \theta)=0 \tag{2}$$

Initial Condition (in the whole space domain Ω_s)

$$e(\boldsymbol{x},\ t=0,\ \phi)=0 \tag{3}$$

Boundary Conditions (on the boundaries S)

$$h(\boldsymbol{X},\ t,\ \phi)=0 \tag{4}$$

Constraints (in the subdomains $(\Omega^g=\Omega_s^g\times\Omega_t^g)$)

$$g(\boldsymbol{x},\ t,\ \phi,\ \theta)\lesseqqgtr 0 \tag{5}$$

in which ϕ=the state variable (temperature, concentration, potential, etc.); θ=the decision variable (controllable load, controllable charge, etc.); $\{\{\phi\}\}$=vector of the state variables in the whole domain Ω; $\{\{\theta\}\}$=vector of the decision variables in the whole domain Ω; $\Omega=\Omega_s\times\Omega_t$=the whole space-time domain; Ω_s=the whole space domain; Ω_t=the whole time domain; $\Omega^g=\Omega_s^g\times\Omega_t^g$=sub-space-time domain; Ω_s^g=sub-space domain; Ω_t^g=sub-time domain; S=the boundary; $\boldsymbol{x}$=Cartesian coordinates $(x,\ y,\ z)$; $\boldsymbol{X}$=Cartesian coordinates of the boundary $(X,\ Y,\ Z)$; and t=time.

3. Formulation of the FD & LP Method

The combined use of finite difference method and linear programming in Eqs. 1-5 yields the following matrix-vector forms of the FD & LP method.

Objective Function

$$Z=\underset{\{\{_j\theta_i^\tau\}\}}{Opt.}\ f(\{\{\phi_n^\tau\}\},\ \{\{_j\theta_i^\tau\}\}) \tag{6}$$

subject to:

Equilibrium Equations ((T×N)-Eqs.)

$$\left[\boldsymbol{A}+\frac{1}{\Delta t^1}\boldsymbol{C}\right]\{\phi_n^1\}+[\boldsymbol{D}^1]\{_j\theta_i^1\}=\{Q_n^1\}+\left[\frac{1}{\Delta t^1}\boldsymbol{C}\right]\{\Phi_n^0\}\ (\tau=1) \tag{7-1}$$

$$-\left[\frac{1}{\Delta t^\tau}\boldsymbol{C}\right]\{\phi_n^{\tau-1}\}+\left[\boldsymbol{A}+\frac{1}{\Delta t^\tau}\boldsymbol{C}\right]\{\phi_n^\tau\}+[\boldsymbol{D}^\tau]\{_j\theta_i^\tau\}=\{Q_n^\tau\}\ (\tau=2\sim T) \tag{7-2}$$

Constraints (L-Eqs.)

$$[\boldsymbol{G}_\phi]\{\{\phi_n^\tau\}\}+[\boldsymbol{G}_\theta]\{\{_j\theta_i^\tau\}\}\lesseqqgtr\{\{B_l\}\} \tag{8}$$

Nonnegative Conditions

$$\phi_n^\tau\geqq 0(\tau=1\sim T,\ n=1\sim N),\ _j\theta_i^\tau\geqq 0\ (\tau=1\sim T,\ i=1\sim I^\tau) \tag{9}$$

in which $[\boldsymbol{A}]$=the state matrix derived from finite difference method, $(N\times N)$ matrix; $[\boldsymbol{D}^\tau]$= the decision matrix at τth time step, $(N\times I^\tau)$ sparse matrix; $[\boldsymbol{G}_\phi]$=the state-constraint matrix, $(L\times(T\times N))$ matrix; $[\boldsymbol{G}_\theta]$=the decision-constraint matrix, $\left(L\times\left(\sum_{\tau=1}^{T}I^\tau\right)\right)$ matrix, $[\boldsymbol{C}]$=the capacity matrix derived from finite difference method, $(N\times N)$ diagonal matrix with unit elements; $\{\phi_n^\tau\}$=vector of the state variables at τth time step; $\{_j\theta_i^\tau\}$=vector of the decision variables (controllable load vector, controllable charge vector, etc.) at τth time step; $\{Q_n^\tau\}$= constant vector (uncontrollable load vector, uncontrollable charge vector, etc.) at τth time step; $\{\Phi_n^0\}$ =the initial state vector; $\{\{B_l\}\}$ =constant vector in the constraints; $\tau=1\sim T=$ time step number; Δt^τ =increment of time in τth time step; $n=1\sim N=$state variable number at each time step, node number in the finite difference meshes; $i=1\sim I^\tau$ =decision variable number at τth time step; j=node number fitted for ith decision variable; and $l=1\sim L$=constraint number.

The FD & LP method is one that optimizes the objective function under the conditions of the equilibrium equations and the constraints. Since all of the variables in linear programming have to be nonnegative because of the limitation in the computational algorithm based on the simplex method, the nonnegative conditions (Eq. 9) are required. In the FD & LP method the number of the variables is $\left(T\times N+\sum_{\tau=1}^{T} I^\tau\right)$, the number of the equilibrium equations is $(T\times N)$, and the number of the constraints is L, respectively. In the sense of general linear programming, the equilibrium equations of the FD & LP method are also the constraints. Thus, the FD & LP method is a kind of linear programming in which the number of the variables is $\left(T\times N+\sum_{\tau=1}^{T} I^\tau\right)$ and the number of the constraints is $(T\times N+L)$, respectively. In the FD & LP method the solution for the state variables and the solution for the decision variables are obtained simultaneously by the simplex method.

The euilibrium equations (Eqs. 7) are obtained by the following procedures. At first, finite difference method is applied to the governing equation (Eq. 2). The application yields the following algebraic equations.

$$[\boldsymbol{A}]\{\phi_n^\tau\}+[\boldsymbol{C}]\left\{\frac{\partial\phi}{\partial t}\Big|_n^\tau\right\}-\{\theta_n^\tau\}=\{Q_n^\tau\}\quad(\tau=1\sim T)\tag{10}$$

Although several time stepping scheme in finite difference method have been presented, the following backward differencing is used in this research.

$$\left\{\frac{\partial\phi}{\partial t}\Big|_n^\tau\right\}=\frac{1}{\Delta t^\tau}(\{\phi_n^\tau\}-\{\phi_n^{\tau-1}\})\quad(\tau=1\sim T)\tag{11}$$

In order to reduce the number of the decision variables, θ_n^τ should be dropped at the nodal points where the controllable load does not exist. Therefore, the following expression for the decision variables is used instead of $\{\theta_n^\tau\}$ and the number of the decision variables at each time step is reduced from N to I^τ.

$$-[\boldsymbol{D}^\tau]\{_j\theta_i^\tau\}=-[d_{ni}^\tau]\{_j\theta_i^\tau\}\tag{12}$$

in which $[\boldsymbol{D}^\tau]=[d_{ni}^\tau]$=the decision matrix, $(N\times I^\tau)$ sparse matrix composed of zero elements with the exceptions of '-1' in I^τ elements whose row number is j and whose column number is i.

Substitution of Eqs. 11 and 12 into Eqs. 10 yields the equilibrium equations. (Eqs. 7).

4. The FD & LP Method in Control of Field Problems

In order to clarify the features of the FD & LP method, the application of the method to field problems (heat conduction, diffusion-convection, electric or magnetic potential, ect.) is studied. In such field problems the basic differential systems are as follows:

Objective Function (throughout the whole domain $(\Omega=\Omega_s\times\Omega_t)$)

$$Z = \underset{\{\{\theta\}\}}{Opt.}\ f(\{\{\phi\}\},\ \{\{\theta\}\}) = \begin{cases} \underset{\{\{\theta\}\}}{Max.}\ f(\{\{\phi\}\},\ \{\{\theta\}\}) \\ \underset{\{\{\theta\}\}}{Min.}\ f(\{\{\phi\}\},\ \{\{\theta\}\}) \end{cases} \tag{13}$$

subject to:

Equilibrium Equations

Governing Differential Equation (in the whole domain $(\Omega = \Omega_s \times \Omega_t)$*)*

$$\underbrace{\omega \frac{\partial \phi}{\partial t} = div(K\ grad\ \phi) - \gamma\phi}_{\phi\text{-terms}} + \underbrace{\theta}_{\theta\text{-term}} + \underbrace{Q}_{const} \tag{14}$$

Initial Condition (in the whole space domain Ω_s*)*

$$\phi(x,\ y,\ t=0) = \Phi^0(x,\ y) \tag{15}$$

Boundary Conditions (on the boundaries S)

$$\phi(X,\ Y,\ t) = \Phi_b(X,\ Y,\ t), \frac{\partial \phi}{\partial n}(X,\ Y,\ t) = 0 \tag{16}$$

Constraints (in the subdomains $(\Omega^g = \Omega_s^g \times \Omega_t^g)$*)*

$$\begin{aligned} &\underline{\Phi} \leq \phi \leq \overline{\Phi},\ or \begin{cases} \phi \geq \underline{\Phi} \\ \phi \leq \overline{\Phi} \end{cases} \\ &\underline{\Theta} \leq \theta \leq \overline{\Theta},\ or \begin{cases} \theta \geq \underline{\Theta} \\ \theta \leq \overline{\Theta} \end{cases} \end{aligned} \tag{17}$$

in which ϕ = the state variable (temperature, concentration, potential, etc.); θ = the decision

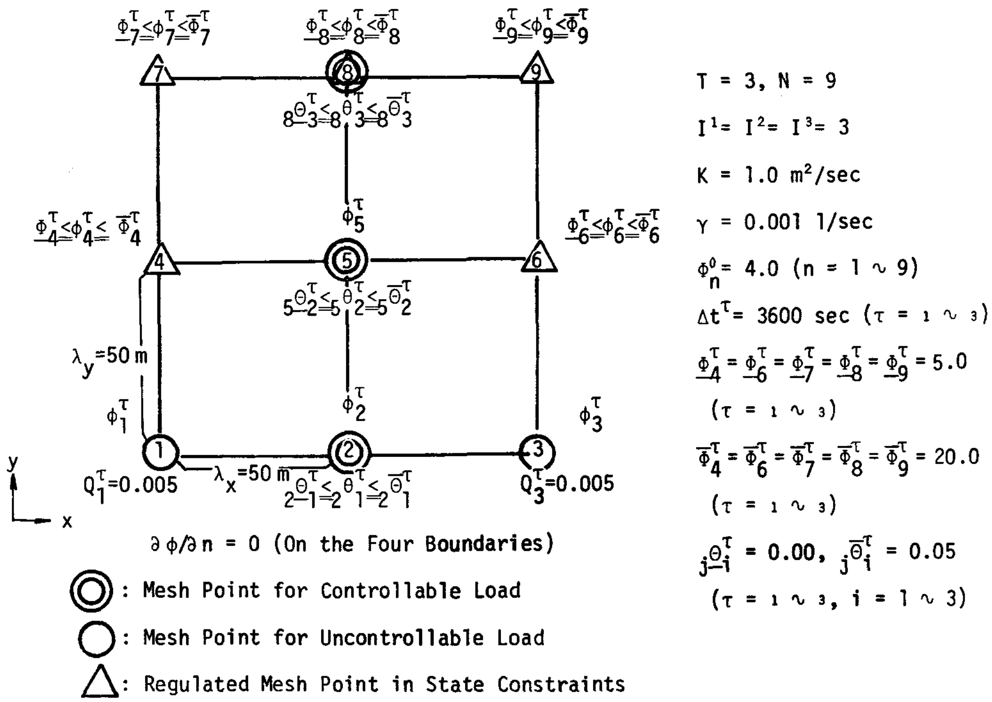

Fig. 3 Simple Model of the FD & LP Method in Field Problem

variable (controllable load, controllable rate of production, controllable charge, etc.); Q=constant (uncontrollable load, uncontrollable rate of production, uncontrollable charge, etc.), K= thermal conductivity, diffusion coefficient, etc.; ω=coefficient ($\omega=\rho c$ in heat conduction equation, $\omega=1$ in diffusion convection equation, etc.); Φ^0=initial state; Φ_b=prescribed boundary value; $\underline{\Phi}$=the lower limit of the state variable; $\bar{\Phi}$=the upper limit of the state variable; $\underline{\Theta}$= the lower limit of the decision variable; $\bar{\Theta}$=the upper limit of the decision variable; and γ= decay factor.

As for the constraints, although only the lower and upper limits of the state variable and decision variable are imposed in the above equation systems, we can impose other constraints, if necessary.

The application of the FD & LP method to the basic differential systems yields the matrix-vector forms. The formulation of the FD & LP method for a simple model (see Fig. 3) in heat conduction problem are shown in Eqs. 18-26. In the model all of the boundaries are nonconvective ones, or $\partial\phi/\partial n=0$ on the all boundaries.

Objective Function

$$Z=\underset{\{\{_j\theta_i^\tau\}\}}{Opt.}\ f(\{\{\phi_n^\tau\}\},\ \{\{_j\theta_i^\tau\}\})=\underset{\{\{_j\theta_i^\tau\}\}}{Opt.}\left(\sum_{\tau=1}^{3}\left(\sum_{n=1}^{9}\alpha_n^\tau\phi_n^\tau+\sum_{i=1}^{3}\beta_i^\tau{}_j\theta_i^\tau\right)\right) \qquad (18)$$

subject to:

Equilibrium Equations ((3×9)-*Eqs.*)

$$\left[A+\frac{1}{\Delta t^1}C\right]\{\phi_n^1\}+[D^1]\{_j\theta_i^1\}=\{Q_n^1\}+\left[\frac{1}{\Delta t^1}C\right]\{\Phi_n^0\}\quad(\tau=1) \qquad (19\text{-}1)$$

$$-\left[\frac{1}{\Delta t^2}C\right]\{\phi_n^1\}+\left[A+\frac{1}{\Delta t^2}C\right]\{\phi_n^2\}+[D^2]\{_j\theta_i^2\}=\{Q_n^2\}\quad(\tau=2) \qquad (19\text{-}2)$$

$$-\left[\frac{1}{\Delta t^3}C\right]\{\phi_n^2\}+\left[A+\frac{1}{\Delta t^3}C\right]\{\phi_n^3\}+[D^3]\{_j\theta_i^3\}=\{Q_n^3\}\quad(\tau=3) \qquad (19\text{-}3)$$

Constraints ((3×(2×(5+3)))-*Eqs.*)

$\phi_4^\tau\geqq\underline{\Phi}_4^\tau(=5.0)$, $\phi_4^\tau\leqq\bar{\Phi}_4^\tau(=20.0)$, $\phi_6^\tau\geqq\underline{\Phi}_6^\tau(=5.0)$, $\phi_6^\tau\leqq\bar{\Phi}_6^\tau(=20.0)$,
$\phi_7^\tau\geqq\underline{\Phi}_7^\tau(=5.0)$, $\phi_7^\tau\leqq\bar{\Phi}_7^\tau(=20.0)$, $\phi_8^\tau\geqq\underline{\Phi}_8^\tau(=5.0)$, $\phi_8^\tau\leqq\bar{\Phi}_8^\tau(=20.0)$,
$\phi_9^\tau\geqq\underline{\Phi}_9^\tau(=5.0)$, $\phi_9^\tau\leqq\bar{\Phi}_9^\tau(=20.0)$, ${}_2\theta_1^\tau\geqq{}_2\underline{\Theta}_1^\tau(=0.00)$, ${}_2\theta_1^\tau\leqq{}_2\bar{\Theta}_1^\tau(=0.05)$,
${}_5\theta_2^\tau\geqq{}_5\underline{\Theta}_2^\tau(=0.00)$, ${}_5\theta_2^\tau\leqq{}_5\bar{\Theta}_2^\tau(=0.05)$, ${}_8\theta_3^\tau\geqq{}_8\underline{\Theta}_3^\tau(=0.00)$, ${}_8\theta_3^\tau\leqq{}_8\bar{\Theta}_3^\tau$ (=0.05)
($\tau=1\sim3$) (20)

Nonnegative Conditions

$\phi_n^\tau\geqq0(\tau=1\sim3,\ n=1\sim9)$, ${}_j\theta_i^\tau\geqq0(\tau=1\sim3,\ i=1\sim3)$ (21)

with

$$[A]=\begin{pmatrix}
0.0026 & -0.0008 & 0 & -0.0008 & 0 & 0 & 0 & 0 & 0\\
-0.0004 & 0.0026 & -0.0004 & 0 & -0.0008 & 0 & 0 & 0 & 0\\
0 & -0.0008 & 0.0026 & 0 & 0 & -0.0008 & 0 & 0 & 0\\
-0.0004 & 0 & 0 & 0.0026 & -0.0008 & 0 & -0.0004 & 0 & 0\\
0 & -0.0004 & 0 & -0.0004 & 0.0026 & -0.0004 & 0 & -0.0004 & 0\\
0 & 0 & -0.0004 & 0 & -0.0008 & 0.0026 & 0 & 0 & -0.0004\\
0 & 0 & 0 & -0.0008 & 0 & 0 & 0.0026 & -0.0008 & 0\\
0 & 0 & 0 & 0 & -0.0008 & 0 & -0.0004 & 0.0026 & -0.0004\\
0 & 0 & 0 & 0 & 0 & -0.0008 & 0 & -0.0008 & 0.0026
\end{pmatrix} \qquad (22)$$

$$[C]=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{23}$$

$$[D^{\tau}]=\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad (\tau=1\sim3) \tag{24}$$

$$\begin{Bmatrix} \Phi_1^0 \\ \Phi_2^0 \\ \Phi_3^0 \\ \Phi_4^0 \\ \Phi_5^0 \\ \Phi_6^0 \\ \Phi_7^0 \\ \Phi_8^0 \\ \Phi_9^0 \end{Bmatrix}=\begin{Bmatrix} 4.0 \\ 4.0 \\ 4.0 \\ 4.0 \\ 4.0 \\ 4.0 \\ 4.0 \\ 4.0 \\ 4.0 \end{Bmatrix} \quad (25) \qquad \begin{Bmatrix} Q_1^{\tau} \\ Q_2^{\tau} \\ Q_3^{\tau} \\ Q_4^{\tau} \\ Q_5^{\tau} \\ Q_6^{\tau} \\ Q_7^{\tau} \\ Q_8^{\tau} \\ Q_9^{\tau} \end{Bmatrix}=\begin{Bmatrix} 0.005 \\ 0 \\ 0.005 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (\tau=1\sim3) \tag{26}$$

The units of the state variable and the decision variable are not described in the simple model, because the FD & LP method is applicable to general physical problems irrespective of the variable considered.

5. Conclusions

The generalization of the FD (Finite Difference) & LP (Linear Programming) method was presented. A formulation of the method in field problems is studied. The tractability not only in the initial and boundary conditions but also in the equality or inequality constraints make sure that the FD & LP method becomes one of the useful techniques for several new types of boundary value problems. In a manner similar to the FD & LP method, the FE (Finite Element) & LP (Linear Programming) method has been developed by the combined use of the finite element method with linear programming. An efficient computational algorithm for the FE & LP method in water pollution control has been presented by Futagami[7]. The related methods of the FD & LP method such as the FD (Finite Difference)

& NLP (Non-Linear Programming) method and the SFD (Stochastic Finite Difference) & LP (Linear Programming) method could be developed.

The developments certainly make it possible to solve more complicated problems.

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