Quadratic Programmings in Paired Spaces

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§ 1. Introduction

Quadratic programmings in finite dimensional spaces are treated by W. S. Dorn [1], M. D. Grigoriadis and K. Ritter [2] and others.

In linear programmings, objective functions are linear functionals and in quadratic programmings, they are quadratic functionals. Here we show that quadratic programmings in infinite spaces are considered and similar theorems as ones in finite spaces are consitituted.

Let X and Y be real linear spaces paired under the bilinear functional $((\ ,\ ))_1$ and Z and W be real linear spaces paired under the bilinear functional $((\ ,\ ))_2$. The weak topology on X is denoted by $w(X, Y)$ and the Mackey topology on X is denoted by $s(X, Y)$. We define a quadratic programming as follows:

$$\text{Minimize } f(x)=((x, Mx))_1+((x, y_0))_1$$
$$\text{subject to } x\in P \text{ and } Ax-z_0\in Q,$$

where M is a linear transformation from X into Y which is $w(X, Y)$-$w(Y, X)$ continuous, symmetric and positive semidefinite, A is a linear transformation X into Z which is $w(X, Y)$-$w(Z, W)$ continuous, P is a convex cone in X which is $w(X, Y)$-closed, Q is a convex cone in Z which is $w(Z, W)$-closed and y_0 and z_0 are elements of Y and Z respectively. We call this program the primal quadratic program. Furthermore, the dual quadratic program to the above primal program is:

$$\text{Maximize } g(u, w)=-((u, Mu))_1+((z_0, w))_2$$
$$\text{subject to } w\in Q^+ \text{ and } y_0-A^*w+2Mu\in P^+,$$

where A^* is the dual transformation of A, $P^+=\{y\in Y;\ ((x, y))_1\geqq 0 \text{ for any } x\in P\}$ and $Q^+=\{w\in W;\ ((z, w))_2\geqq 0 \text{ for any } z\in Q\}$. We note that the dual transformation of M is identified with M by its symmetry.

In general, $\inf\{f(x);\ x\in P \text{ and } Ax-z_0\in Q\}\geqq \sup\{g(u, w);\ w\in Q^+ \text{ and } y_0-A^*w+2Mu\in P^+\}$, therefore if $x_0\in P$ with $Ax_0-z_0\in Q$, $(u_0, w_0)\in X\times Q^+$ with $y_0-A^*w_0+2Mu_0\in P^+$ and if $f(x_0)=g(u_0, w_0)$, then $f(x_0)=\inf\{f(x);\ x\in P \text{ and } Ax-z_0\in Q\}=$

$g(u_0, w_0)=\sup\{g(u, w);\ w\in Q^+ \text{ and } y_0-A^*w+2Mu\in P^+\}$.

§ 2. Duality theorems for the quadratic programmings

We shall recall the theory of infinite linear programmings studied in [5].

X, Y, Z, W, P, Q, A, y_0 and z_0 are same as we mentioned in § 1. A primal linear program is:

Minimize $((x, y_0))_1$
subject to $x\in P$ and $Ax-z_0\in Q$,

and the dual linear program is:

Maximize $((z_0, w))_2$
subject to $w\in Q^+$ and $y_0-A^*w\in P^+$.

Now, let R be the set of real numbers, R_0 be the set of non-negative real numbers and $X\times R$ and $Y\times R$ be paired spaces under the bilinear functional $((x, y))_1+rs$ for $(x, r)\in X\times R$ and $(y, s)\in Y\times R$.

Kretschmer proved in [5]

THEOREM. *Suppose that there exists an $x\in P$ such that $Ax-z_0\in Q$ and* $\inf\{((x, y_0))_1;\ x\in P \text{ and } Ax-z_0\in Q\}$ *is finite. If $G=\{(A^*w+y, r-((z_0, w))_2); y\in P^+, w\in Q^+$ and $r\in R_0\}$ is $w(Y\times R, X\times R)$-closed, then there exists a $w_0\in Q^+$ such that $y_0-A^*w_0\in P^+$ and* $\inf\{((x, y_0))_1;\ x\in P \text{ and } Ax-z_0\in Q\}=\sup\{((z_0, w))_2;\ w\in Q^+ \text{ and } y_0-A^*w\in P^+\}=((z_0, w_0))_2$.

We can apply this theorem to the duality theorems for quadratic programmings.

THEOREM 1. *Let x_0 be a solution of the primal quadratic program* i.e., $x_0\in P$, *$Ax_0-z_0\in Q$ and $f(x_0)=inf\{f(x);\ x\in P$ and $Ax-z_0\in Q\}$. If G is $w(Y\times R, X\times R)$-closed, then there exists w_0 such that (x_0, w_0) is a solution of the dual quadratic program and $f(x_0)=g(x_0, w_0)$ i. e., $w_0\in Q^+$, $y_0-A^*w_0+2Mx_0\in P^+$ and $g(x_0, w_0)=sup\{g(u, w);w\in Q^+$ and $y_0-A^*w+2Mu\in p^+\}=f(x_0)$.*

PROOF. We consider the following linear program.

Minimize $F(x)=((x, 2Mx_0))_1+((x, y_0))_1-((x_0, Mx_0))_1$
subject to $x\in P$ and $Ax-z_0\in Q$.

Then we can prove that x_0 is a solution of the above program.
In fact, let there exist an $x^*\in P$ such that $Ax^*-z_0\in Q$ and $F(x^*)<F(x_0)$ i. e., $((x^*-x_0, 2Mx_0+y_0))_1<0$.
Hence, if we set $x=x_0+k(x^*-x_0)$, then $x\in P$, $Ax-z_0\in Q$ and $f(x)-f(x_0)=k\{((x^*-$

$x_0,\ 2Mx_0+y_0))_1+k((x^*-x_0,\ M(x^*-x_0)))_1\}<0$ for sufficiently small $k>0$. It contradicts that x_0 is a minimizing solution. Now, we have the dual program to the above:

$$\begin{aligned}&\text{Maximize } ((z_0,\ w))_2-((x_0,\ Mx_0))_1\\&\text{subject to } w\in Q^+ \text{ and } y_0-A^*w+2Mx_0\in P^+.\end{aligned}$$

From the Kretschmer's theorem, we obtain a w_0 such that $w_0\in Q^+$, $y_0-A^*w_0+2Mx_0\in P^+$ and $((z_0,\ w_0))_2=((x_0,\ 2Mx_0))_1+((x_0,\ y_0))_1$ i. e., $f(x_0)=g(x_0,\ w_0)$. This implies that $(x_0,\ w_0)$ is a maximizing solution of the dual quadratic program.

REMARK. If $x_0\in P$ and $Ax_0-z_0\in Q$, then a necessary and sufficient condition in order that $((x_0,\ Mx_0))_1+((x_0,\ y_0))_1=\inf\{((x,\ Mx))_1+((x,\ y_0))_1;\ x\in P$ and $Ax-z_0\in Q\}$ is that $((x_0,\ 2Mx_0+y_0))_1=\inf\{((x,\ 2Mx_0+y_0))_1;\ x\in P$ and $Ax-z_0\in Q\}$.

Now, we note that the dual quadratic program is equivalent to the following program:

$$\begin{aligned}&\text{Minimize } (((u,\ w),\ \bar{M}(u,\ w)))+(((u,\ w),\ (0,\ -z_0)))\\&\text{subject to } (u,\ w)\in X\times Q^+ \text{ and } B(u,\ w)-y_0\in -P^+,\end{aligned}$$

where $((\ ,\ ))$ is a pairing between $X\times W$ and $Y\times Z$ defined by $(((x,\ w),\ (y,\ z)))=((x,\ y))_1+((z,\ w))_2$ for $(x,\ w)\in X\times W$ and $(y,\ z)\in Y\times Z$, $\bar{M}(u,\ w)=(Mu,\ 0)$ for $(u,\ w)\in X\times W$ and $B(u,\ w)=A^*w-2Mu$ for $(u,\ w)\in X\times W$. It is evident that this program is a quadratic programming and its dual program is:

$$\begin{aligned}&\text{Maximize } -((u,\ Mu))_1+((x,\ y_0))_1\\&\text{subject to } x\in -P,\ M(u+x)=0 \textit{ and } -z_0-Ax\in Q.\end{aligned}$$

THEOREM 2. *Let $(u_0,\ w_0)$ be a maximizing solution of the dual quadratic program, then there exists a minimizing solution $\bar{x}$ of the primal quadratic program such that $M\bar{x}=Mu_0$ and $f(\bar{x})=g(u_0,\ w_0)$ if $G'=\{(-2Mx,\ Ax+z,\ r-((x,\ y_0))_1);\ x\in -P,\ z\in Q$ and $r\in R_0\}$ is $w(Y\times Z\times R,\ X\times W\times R)$-closed.*

PROOF. From THEOREM 1, there exists an x_0 such that $(u_0,\ x_0)$ is a maximizing solution of the dual quadratic program mentioned just above i. e., $x_0\in -P$, $M(u_0+x_0)=0$, $-z_0-Ax_0\in Q$ and $((u_0,\ Mu_0))_1-((x_0,\ y_0))_1=-((u_0,\ Mu_0))_1+((z_0,\ w_0))_2$. We shall show that $\bar{x}=-x_0$ is a minimizing solution of the primal quadratic program. To see this, it is sufficient to show that $((\bar{x},\ M\bar{x}))_1+((\bar{x},\ y_0))_1=-((u_0,Mu_0))_1+((z_0,w_0))_2$.

$$\begin{aligned}&((\bar{x},\ M\bar{x}))_1+((\bar{x},\ y_0))_1+((u_0,\ Mu_0))_1-((z_0,\ w_0))_2\\&=((\bar{x},\ Mu_0))_1+((u_0,\ Mu_0))_1+((\bar{x},\ y_0))_1-((z_0,\ w_0))_2\\&=((\bar{x}+u_0,\ Mu_0))_1-2((u_0,\ Mu_0))_1=((\bar{x}-u_0,\ Mu_0))_1\\&=((u_0,\ M(\bar{x}-u_0)))_1=0.\end{aligned}$$

REMARK. *G' is $w(Y\times Z\times R,\ X\times W\times R)$*-closed and $\bar{x}=u_0$ if M is onto.

LEMMA. *Let* $\{x_n\}$ *be a sequence such that* $x_n \in P$, $Ax_n - z_0 \in Q$ *for any* $n=1, 2, 3, \dots$, *and* $\{x_n\}$ *converges to an* $\bar{x}$ *in* $w(X, Y)$ *and* $\{((x_n, y_0))_1 + ((x_n, Mx_n))_1\}$ *converges to inf* $\{((x, Mx))_1 + ((x, y_0))_1;\ x \in P$ *and* $Ax - z_0 \in Q\} = \alpha$ *as* n *tends to infinity. Then* $\bar{x} \in P$, $A\bar{x} - z_0 \in Q$ *and* $((\bar{x}, M\bar{x}))_1 + ((\bar{x}, y_0))_1 = \alpha$.

PROOF. Since the set $\{x \in P;\ Ax - z_0 \in Q\}$ is $w(X, Y)$-closed, $\bar{x}$ belongs to this set. Since $\{x_n\}$ converges to $\bar{x}$ in $w(X, Y)$, $\lim_{n\to\infty}((x_n, y_0))_1$ exists and hence $\lim_{n\to\infty}((x_n, Mx_n))_1$ exists. In general, $((\bar{x}, M\bar{x}))_1 + ((\bar{x}, y_0))_1 \geq \alpha = \lim_{n\to\infty}((x_n, Mx_n))_1 + \lim_{n\to\infty}((x_n, y_0))_1 = \lim_{n\to\infty}((x_n, Mx_n))_1 + ((\bar{x}, y_0))_1$ i. e., $((\bar{x}, M\bar{x}))_1 \geq \lim_{n\to\infty}((x_n, Mx_n))_1$. On the other hand, $((x_n - \bar{x}, M(x_n - \bar{x})))_1 \geq 0$, hence $((x_n, Mx_n))_1 \geq ((\bar{x}, Mx_n))_1 + ((x_n, M\bar{x}))_1 - ((\bar{x}, M\bar{x}))_1$. Letting n tend to infinity, we have $\lim_{n\to\infty}((x_n, Mx_n))_1 \geq ((\bar{x}, M\bar{x}))_1$. Consequently $\lim_{n\to\infty}((x_n, Mx_n))_1 = ((\bar{x}, M\bar{x}))_1$, i. e., $\alpha = ((\bar{x}, M\bar{x}))_1 + ((\bar{x}, y_0))_1$.

§ 3. A class of continuous quadratic programming problems

In this section we shall apply the theory of infinite quadratic programmings studied in § 2 to the investigation of the continuous programming problems. Continuous non-linear programmings are investigated by R. C. Grinold [3], M. A. Hanson [4] and many other authors.

For the definition of this section, see [6]. Let

$B(t) = (b_{ij}(t))$ be an $n \times m$ matrix on $[0, T]$,
$D(t) = (d_{ij}(t))$ be an $n \times n$ matrix on $[0, T]$,
$c(t) = (c_i(t))$ be an n vector on $[0, T]$,
$a(t) = (a_j(t))$ be an m vector on $[0, T]$ and
$K(t, s) = (k_{ij}(t, s))$ be an $n \times m$ matrix on $[0, T] \times [0, T]$,

where $0 < T < \infty$, $b_{ij}(t)$, $d_{ij}(t)$, $c_i(t)$, $a_j(t)$ and $k_{ij}(t, s)$ are bounded real-valued functions which are measurable with respect to the Lebesgue measures on the real line and plane respectively, and $D(t)$ is symmetric and positive semidefinite.

Denote $\{x(t) = (x_i(t));\ x(t)$ is an n vector on $[0, T]$, $x_i(t)$ is bounded measurable and $x_i(t) \geq 0$ on $[0, T]$ and $x(t)B(t) - a(t) - \int_t^T x(s)K(s, t)ds \geq 0\}$ by S and $\{(u(t), w(t));\ u(t) = (u_i(t))$ is an n vector on $[0, T]$, $u_i(t)$ is bounded measurable on $[0, T]$, $w(t) = (w_j(t))$ is an m vector on $[0, T]$, $w_j(t)$ is bounded measurable and $w_j(t) \geq 0$ on $[0, T]$ and $c(t) - B(t)w(t) + \int_0^t K(t, s)w(s)ds + 2D(t)u(t) \geq O\}$ by S'.

Consider the programs:

$$\text{Minimize } f(x) = \int_0^T x(t)c(t)dt + \int_0^T x(t)D(t)x(t)dt$$

subject to $x(t) \in S$,

Maximize $g(u, w) = \int_0^T w(t)a(t)dt - \int_0^T u(t)D(t)u(t)dt$

subject to $(u(t), w(t)) \in S'$.

We always assume the following conditions:

(N. 1) $c(t) \geqq 0$ *and* $K(t, s) \geqq 0$.

(N. 2) *There exists* $\beta > 0$ *such that for each* i, j *and* t *either* $b_{ij}(t) = 0$ *or else* $b_{ij}(t) \geqq \beta$. *Also for each* t *and* j, *there exists* $i_j = i_j(t)$ *such that* $b_{i_j j}(t) \geqq \beta$.

Our theorem is:

THEOREM 3. There exist an n vector $x_0(t) \in S$ and an m vector $w_0(t)$ such that $(x_0(t), w_0(t)) \in S'$ and $f(x_0) = \inf\{f(x); x(t) \in S\} = g(x_0, w_0) = \sup\{g(u, w); (u(t), w(t)) \in S'\}$.

PROOF. To apply the THEOREM 1, we choose

$X = Y = L^2_n [0, T]$, $Z = W = L^2_m [0, T]$,

$((x, y))_1 = \int_0^T x(t)y(t)dt$ for $x \in X$ and $y \in Y$,

$((z, w))_2 = \int_0^T z(t)w(t)dt$ for $z \in Z$ and $w \in W$,

$P = \{x \in X; x(t) \geqq 0$ *a. e. on* $[0, T]\}$,

$Q = \{z \in Z; z(t) \geqq 0$ *a. e. on* $[0, T]\}$,

$y_0 = c$, $z_0 = a$ and

$Ax(t) = x(t)B(t) - \int_t^T x(s)K(s, t)ds$,

where $L^2_n [0, T]$ is n product of $L^2 [0, T]$, the space of all real-valued functions on $[0, T]$ which are square integrable and "a. e." implies "almost everywhree with respect to the Lebesgue measure on the line". It is easy to show that $P^+ = P$, $Q^+ = Q$ and $A^*w(t) = B(t)w(t) - \int_0^t K(t, s)w(s)ds$. Denote $\{x \in P; Ax - a \in Q\}$ by S_1 and $\{(u, w) \in X \times Q^+; c - A^*w + 2Du \in P^+\}$ by S_1'. Then $S_1 \neq \phi$, $\inf\{f(x); x \in S_1\}$ is finite by (N. 1) and (N. 2), furtheremore $\inf\{f(x); x \in S\} = \inf\{f(x); x \in S_1\}$ by the same arguments as in THEOREM 4 in [6], in general, $\sup\{g(u, w); (u, w) \in S'\} \leqq \sup\{g(u, w); (u, w) \in S_1'\}$ and G is $w(Y \times R, X \times R)$-closed by THEOREM 2 in [6]. Consequently, to prove THEOREM 3, it is sufficient to show the existence of $x_0 \in S$ such that $f(x_0) = \inf\{f(f); x \in S\}$.

Since $\inf\{f(x); x \in S\}$ is finite, there exists a sequence $\{x^{(k)}\}$ in S such that $\{f(x^{(k)})\}$ converges to $\inf\{f(x); x \in S\}$ as k tends to infinity. By (N. 1) and (N. 2), we may assume that $\{x^{(k)}\}$ is $L^2_n [0, T]$-bounded and therefore we can find a $w(X, Y)$-convergent subsequence of $\{x^{(k)}\}$. Denote its limit by x_0. From the LEMMA in § 2, x_0 is that we want. Though x_0 and w_0 taken by THEOREM 1 are in $L^2_n [0, T]$ and L^2_m

[0, T] respectively, we may assume that x_0 is bounded and hence w_0 is also bounded.

COROLLARY. $\sup\{g(u, w); (u, w)\in S'\}=\sup\{g(u, w); (u, w)\in S_1'\}$.

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