Finite Element & Linear Programming Method in Optimal Structural Systems

Tanehiro FUTAGAMI*, Hidetoshi ITO* and Yuzo SUGA*

(Received Oct. 9, 1980)

Abstract

By combining the finite element method with linear programming, finite element & linear programming method (the FE&LP method) has been developed and systematized in order to solve optimal systems of differential equations with both equality or inequality constraints and an objective function. Such systems are frequently encountered in various engineering and scientific problems of control and optimal design. In this study the application of the FE&LP method to optimization of combined spring systems is presented. The method makes it possible to obtain not only the optimal controllable loads to meet with the imposed upper limits of stresses, but also the nodal displacements in spring systems. The method may become one of the most useful techniques in optimal design of structural systems.

1. Introduction

Finite element & linear programming method (the FE&LP method)[1,2,3,4] has been developed in order to control systems of partial differential equations with both equality or inequality constraints and an objective function. Such systems are frequently encountered in various engineering and scientific problems of control and optimal design. In the development of the FE&LP method, the combined use of finite element method and linear programming has been adopted. The finite element method, originated in structural mechanics, is powerful numerical method for the solution of differential equations because of its generality with respect to geometry and material properties[5,6,7]. Linear programming is one of the most frequently used mathematical methods of operations research[8,9]. In the development of the FE&LP method the concepts of the decision variable and the state variable are adopted as in Bellman's dynamic programming[10,11] and Pontryagin's maximum principle [12,13]. The FE&LP method utilizes the advantages of the numerical techniques of both finite element method and linear programming. Aguado and Remson[14] have suggested the combined use of finite element method with linear programming in the study of ground-water management, in which finite difference method is used instead of finite element method.

In this paper the application of the FE&LP method in optimal structural systems is presented. Several numerical examples of the FE&LP method are studied in combined spring systems. In the examples, the state variable is the nodal displacement at each spring joint and the decision variable is the controllable load at each spring joint. As for the constraints the

* Department of Civil Engineering

upper limit of the stress (allowable stress) in each spring element is imposed. Although the objective function may be composed of the state variables (vector of displacements) and the decision variables (vector of controllable loads), the maximization of total of the decision variables is considered in the examples for the simplicity.

2. Optimization of Elastic Structural Systems by the FE&LP Method

Generally, structural systems are to be designed to meet with several requirements such as cost, weight, strengh, deformation, etc. Much significant optimization problems considered these requirements have been studied by using various mathematical techniques[15,16].

In this study the FE&LP method is applied in order to optimize the following elastic structural systems (see Fig. 1).

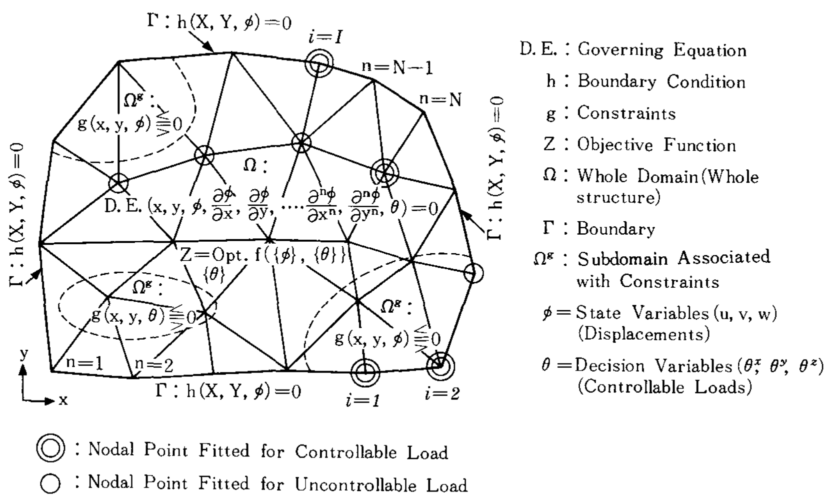


Fig. 1 General Concepts of Finite Element & Linear Programming Method in Optimal Structural Systems

Objective Function (throughout the whole domain Ω*)*

$$Z=Opt.\ f(\{\phi\},\ \{\theta\})=\begin{cases} \underset{\{\theta\}}{Max.}\ f(\{\phi\},\ \{\theta\}) \\ \underset{\{\theta\}}{Min.}\ f(\{\phi\},\ \{\theta\}) \end{cases} \tag{1}$$

subject to:

Equilibrium Equations

Governing Differential Equations (in the whole domain Ω*)*

$$\left.\begin{aligned} &\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+\theta^x+F_x=0, \\ &\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}+\theta^y+F_y=0, \\ &\frac{\partial\tau_{zx}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+\theta^z+F_z=0. \end{aligned}\right\} \tag{2}$$

Displacement-Strain Equations (in the whole domain Ω*)*

$$\left.\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x}, \quad \gamma_{xy}=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y},\\
\varepsilon_y &= \frac{\partial v}{\partial y}, \quad \gamma_{yz}=\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z},\\
\varepsilon_z &= \frac{\partial w}{\partial z}, \quad \gamma_{zx}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}.
\end{aligned}\right\} \tag{3}$$

Stress-Strain Equations (in the whole domain Ω)

$$\{\sigma\}=[S]\{\varepsilon\},\ or,\quad \left.\begin{aligned}
\sigma_x &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left\{\varepsilon_x+\frac{\nu}{1-\nu}(\varepsilon_y+\varepsilon_z)\right\},\\
\sigma_y &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left\{\varepsilon_y+\frac{\nu}{1-\nu}(\varepsilon_z+\varepsilon_x)\right\},\\
\sigma_z &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left\{\varepsilon_z+\frac{\nu}{1-\nu}(\varepsilon_x+\varepsilon_y)\right\},\\
\tau_{xy} &= \frac{E}{2(1+\nu)}\gamma_{xy}, \quad \tau_{yz}=\frac{E}{2(1+\nu)}\gamma_{yz},\\
\tau_{zx} &= \frac{E}{2(1+\nu)}\gamma_{zx}.
\end{aligned}\right\} \tag{4}$$

Boundary Conditions (on the Boundary Γ)

$$h(X,\ Y,\ Z,\ \phi)=0 \tag{5}$$

Constraints (in the sub-domains Ω^g)

$$g(x,\ y,\ z,\ \phi,\ \theta)\lesseqgtr 0 \tag{6}$$

where

$$[S]=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix}
1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0\\
\frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0\\
\frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 & 0\\
0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0\\
0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)}
\end{bmatrix} \tag{7}$$

$$\{\sigma\}=\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{Bmatrix},\quad \{\varepsilon\}=\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{Bmatrix},\quad \{\phi\}=\begin{Bmatrix}u\\ v\\ w\end{Bmatrix},\quad \{\theta\}=\begin{Bmatrix}\theta^x\\ \theta^y\\ \theta^z\end{Bmatrix} \tag{8}$$

in which f=objective function; ϕ=the state variables; i.e., displacements in x, y and z directions $(u,\ v,\ w)$; θ=the decision variables, i.e., controllable loads in x, y and z directions $(\theta^x,\ \theta^y,\ \theta^z)$; F_x, F_y, F_z=constants=uncontrollable loads in x, y and z directions; σ_x, σ_y, σ_z= normal components of stress parallel to x, y and z axes; τ_{xy}, τ_{yz}, τ_{zx}=shearing stress components in rectangular coordinates; ε_x, ε_y, ε_z=unit elongations in x, y and z directions;

γ_{xy}, γ_{yz}, γ_{zx} = shearing strain components in rectangular coordinates; E = modulus of elaticity; and ν = Poisson's ratio.

The finite element method is used in order to discretize the above-mentioned systems as systems of linear algebraic equations. Then the following matrix-vector forms of the FE&LP method are obtained and the application of linear programming is possible.

Objective Function

$$Z = \underset{\{\theta\}}{Opt.}\ f(\{\phi\}, \{\theta\}) \tag{9}$$

subject to:

Equilibrium Equations

$$[A]\{\phi\} + [D]\{\theta\} = \{f\} \tag{10}$$

Constraints

$$[G_\phi]\{\phi\} + [G_\theta]\{\theta\} \gtreqless \{b\} \tag{11}$$

in which $[\boldsymbol{A}]$ = the state matrix = the stiffness matrix; $[\boldsymbol{D}]$ = the decision matrix; $[\boldsymbol{G}_\phi]$ = the state-constraint matrix; $[\boldsymbol{G}_\theta]$ = the decision-constraint matrix; $\{\phi\}$ = the state vector = vector of the nodal displacements in the whole finite elements; $\{\theta\}$ = the decision vector = vector of the controllable loads; $\{f\}$ = constant vector associated with uncontrollable nodal loads; $\{b\}$ = constant vector associated with constraints.

3. Optimization of Combined Spring Systems by the FE&LP Method

In order to clarify the features of the FE&LP method, the method applied to the following structural systems in an assembly of one-dimensional elastic bars (see Fig. 2).

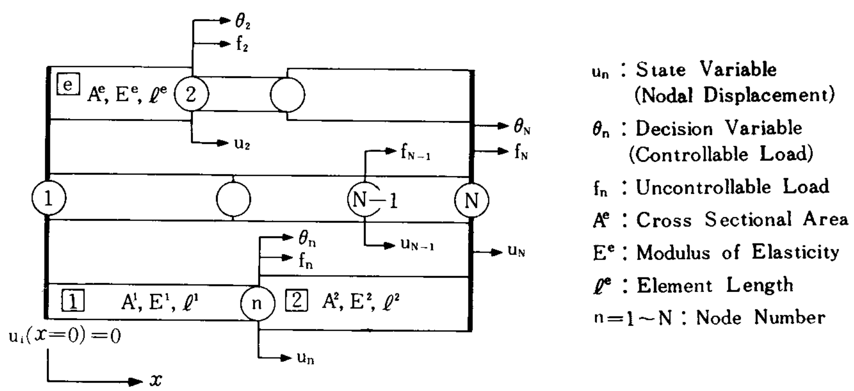


Fig. 2 Assembly of One-Dimensional Bars Divided into Finite Elements

Objective Function (throughout the whole elastic bars Ω)

$$Z = \underset{\{\theta\}}{Opt.}\ f(\{u\}, \{\theta\}) = \underset{\{\theta\}}{Max.}\ f(\{\theta\}) \tag{12}$$

subject to:

Equilibrium Equations

Governing Differential Equations (in the whole elastic bars Ω)

$$\frac{d\sigma}{dx} + \theta + F = 0 \tag{13}$$

Displacement-Strain Equation (in the whole elastic bars Ω)

$$\varepsilon = \frac{du}{dx} \tag{14}$$

Stress-Strain Equation (in the whole elastic bars Ω)

$$\sigma = E\varepsilon \tag{15}$$

Boundary Conditions (on the boundary Γ)

$$u(x=0)=0 \tag{16}$$

Constraints (in the whole elastic bars Ω)

$$\sigma \leqq \bar{\sigma},\ or,\ \frac{du}{dx}E \leqq \bar{\sigma} \tag{17}$$

Nonnegative Conditions

$$u \geqq 0,\ \theta \geqq 0 \tag{18}$$

in which u=the state variable (displacement in x direction); θ=the decision variable (controllable load in x direction); x=coordinate; F=constant=uncontrollable load in x direction; σ=normal stress in x direction; ε=strain=unit elongation in x direction; E=modulus of elasticity; $\bar{\sigma}$=upper limit of the stress (allowable stress).

Substituting Eq. (15) for σ and Eq. (14) for ε into Eq. (13), we obtain the following governing differential equation.

$$E\frac{d^2u}{dx^2}+\theta+F=0 \tag{19}$$

In this research, Galerkin finite element method is used in order to discretize the above equation. As for the details of Galerkin finite element method, one may follow Zienkievicz[5].

The assembly of one-dimensional elastic bars is divided into small regions called finite elements. The central idea common to all varieties of the finite element method is the description of the variation of unknown (the state variable u for the present instance) by shape functions in each element. The description is given by the following approximation (see Fig. 3).

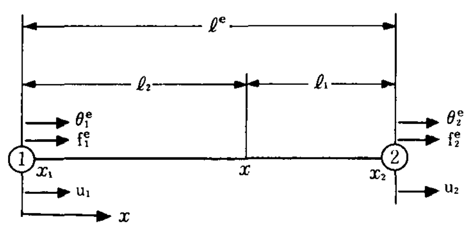


Fig. 3 Line Element

$$u=[N]\{u\}^e=[N_1 N_2]\begin{Bmatrix}u_1\\u_2\end{Bmatrix} \tag{20}$$

in which $[N]=[N_1\ N_2]$=the usual shape functions expressed by line coordinates (ζ_1 and ζ_2). The line coordinates are given by the following equations (see Fig. 3).

$$\begin{aligned}\zeta_1&=l_1/l^e=(x_2-x)/l^e=a_1+b_1x\\ \zeta_2&=l_2/l^e=(x-x_1)/l^e=a_2+b_2x\\ \zeta_1&+\zeta_2=1\end{aligned} \tag{21}$$

where

$$l^e = x_2 - x_1,\ a_1 = x_2/l^e,\ b_1 = -1/l^e,\ a_2 = -x_1/l^e,\ b_2 = 1/l^e$$

Using the weighted residual process for the governing equation (Eq. 19), we obtain the following equation.

$$\int_{l^e} N_n \left[E\frac{d^2u}{dx^2} + \theta + F \right] A^e dx = 0 \tag{22}$$

Integration by part of the above equation and substitution of Eqs. 20, 21 yields the following stiffness equation in each line element.

$$[K]^e\{u\}^e = \{\theta\}^e + \{f\}^e$$

, or,

$$\begin{bmatrix} k^e & -k^e \\ -k^e & k^e \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} \theta_1{}^e \\ \theta_2{}^e \end{Bmatrix} + \begin{Bmatrix} f_1{}^e \\ f_2{}^e \end{Bmatrix} \tag{23}$$

where

$k^e = \dfrac{A^e E^e}{l^e}$ (k^e=spring constant; A^e=cross sectional area; l^e=element length; E^e=modulus of elasticity)

Substitution of the contribution from the all elements yields the following global stiffness equation.

$$\sum_{p=1}^{N} k_{np} u_p - \theta_n = f_n \quad (n = 1 \sim N)$$

, or,

$$[\underset{N\times N}{\boldsymbol{K}}]\{u_n\} - \{\theta_n\} = \{f_n\} \tag{24}$$

On the boundary $x=0$ with prescribed boundary value of $u(x=0)=0$ (Eq. 16), the equilibrium equation is corrected as follows:

$$k_{nn}u_n = 0,\ or,\ 1 \times u_n = 0 \tag{25}$$

In order to reduce the number of the decision variables from N to I, θ_n in Eq. 24 should be dropped at the nodal point where the controllable load could not exist. Then, Eq. 24 is replaced by the following equation.

$$[\boldsymbol{K}]\{u_n\} + [\boldsymbol{D}]\{{}_j\theta_i\} = \{f_n\} \tag{26}$$

The decision matrix $[\boldsymbol{D}]$ in Eq. (26) is composed of zero elements with the exceptions of '-1' in I elements whose row number is j and whose column number is i (see Figs. 4, 5, 6 and Eq. 33).

Substituting Eqs. (14), (15) into Eq. (17) in Fig. 3, we obtain the following constraint associated with the allowable stress.

$$-u_1 + u_2 \leqq \frac{l^e}{E^e} \bar{\sigma}^e \tag{27}$$

Thus, the discretized equation systems of the FE&LP method in an assembly of bars is obtained as follows:

Objective Function

$$Z = \underset{\{_j\theta_i\}}{Opt.} f(\{u_n\}, \{{}_j\theta_i\}) \simeq Max. \sum_{i=1}^{I} {}_j\theta_i \tag{28}$$

subject to:

Equilibrium Equations (N-Eqs.)

$$\underset{N\times N}{[\boldsymbol{K}]}\{u_n\}+\underset{N\times I}{[\boldsymbol{D}]}\{{}_j\theta_i\}=\{f_n\} \quad (29)$$

Constraints (L-Eqs.)

$$\underset{L\times N}{[\boldsymbol{G}_\phi]}\{u_n\} \leqq \left\{\frac{l^e}{E^e}\bar{\sigma}^e\right\} \quad (30)$$

Nonnegative Conditions

$$u_n \geqq 0 \ (n=1\sim N),\ {}_j\theta_i \geqq 0 \ (i=1\sim I) \quad (31)$$

In this stage, we can find that the assembly of bars shown in Fig. 2 is replaced by combined spring systems as shown in Fig. 4.

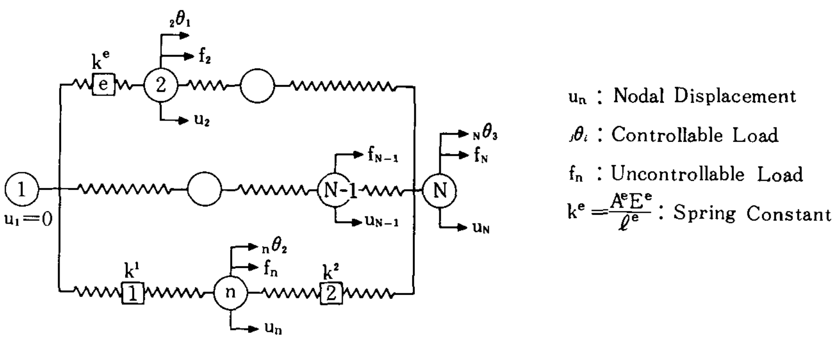


Fig. 4 Combined Spring Systems Corresponding to Assembly of Bars

4. Numerical Examples

Numerical examples of the FE&LP method are conducted in three different assemblies of bars as shown in Figs. 5 (a), (b), (c). Each assembly of bars is replaced by combined

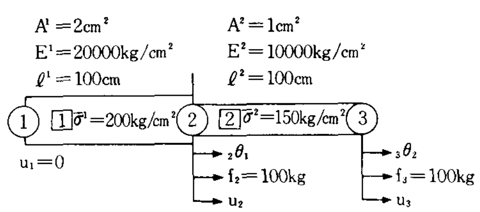


Fig. 5 (a) Assembly of Bars in Example 1

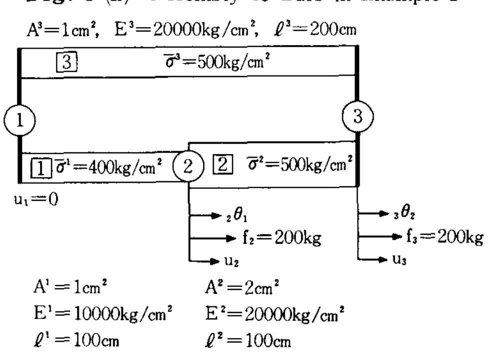


Fig. 5 (b) Assembly of Bars in Example 2

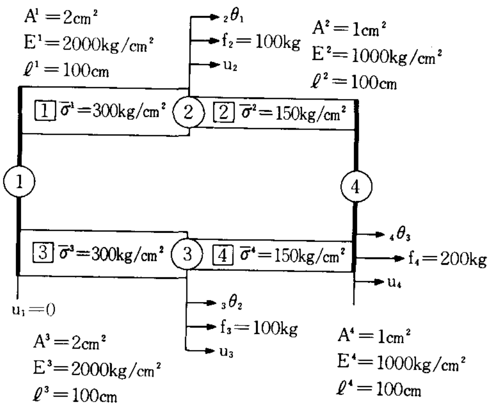


Fig. 5 (c) Assembly of Bars in Example 3

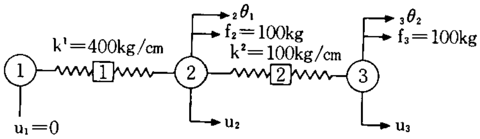


Fig. 6 (a) Combined Spring Systems in Example 1

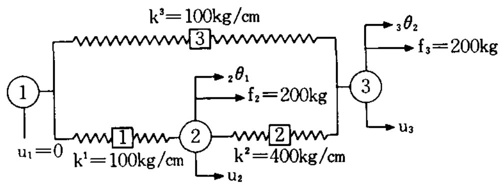


Fig. 6 (b) Combined Spring Systems in Example 2

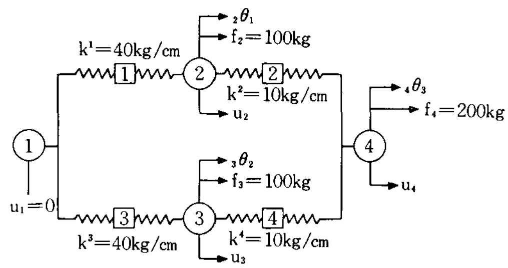


Fig. 6 (c) Combined Spring Systems in Example 3

spring systems as shown in Figs. 6 (a), (b), (c).

In Example 1, for example, the matrix-vector forms of the FE&LP method are as follows:

Objective Function

$$Z = Max. \sum_{i=1}^{2} {}_j\theta_i = Max. \ ({}_2\theta_1 + {}_3\theta_2) \tag{32}$$

subject to:

Equilibrium Equations (3-*Eqs.*)

$$\begin{bmatrix} 1 & 0 & 0 \\ -k^1 & k^1+k^2 & -k^2 \\ & \boldsymbol{K} & \\ 0 & -k^2 & k^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ & \boldsymbol{D} \\ 0 & -1 \end{bmatrix} \begin{Bmatrix} {}_2\theta_1 \\ {}_3\theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f_2 \\ f_3 \end{Bmatrix} \tag{33}$$

Constraints (2-*Eqs.*)

$$\begin{bmatrix} -1 & 1 & 0 \\ & \boldsymbol{G}_\phi & \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \leqq \begin{Bmatrix} l^1\bar{\sigma}^1/E^1 \\ l^2\bar{\sigma}^2/E^2 \end{Bmatrix} \tag{34}$$

Nonnegative Conditions

$$u_1 \geqq 0,\ u_2 \geqq 0,\ u_3 \geqq 0,\ {}_2\theta_1 \geqq 0,\ {}_3\theta_2 \geqq 0 \tag{35}$$

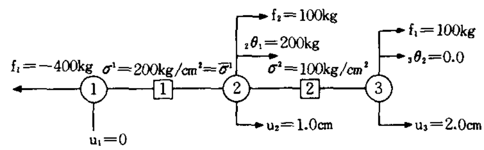
Table 1(a) Input Data on Elements

Example	Element Number	Sectional Area A^e (cm²)	Modulus of Elasticity E^e(kg/cm²)	Element Length l^e (cm)	Allowable Stress ($\bar{\sigma}^e$ kg/cm²)
Example 1	1	2	20,000	100	200
	2	1	10,000	100	150
Example 2	1	1	10,000	100	400
	2	2	20,000	100	500
	3	1	20,000	100	500
Example 3	1	2	2,000	100	300
	2	1	1,000	100	150
	3	2	2,000	100	300
	4	1	1,000	100	150

Table 1(b) Input Data on Nodes

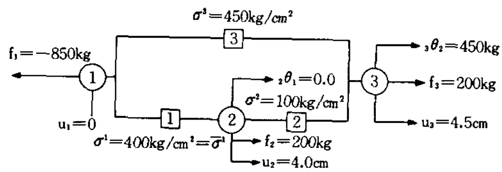
Example	Node Number	Prescribed Displacement	Uncontrollable Load f_n (kg)
Example 1	①	$u_1=0$	
	②		100
	③		100
Example 2	①	$u_1=0$	
	②		200
	③		200
Example 3	①	$u_1=0$	
	②		100
	③		100
	④		200

The input data on these examples are shown in Tables 1 (a), (b). The results are shown in Tables 2 (a), (b) and Figs. 7 (a), (b), (c).



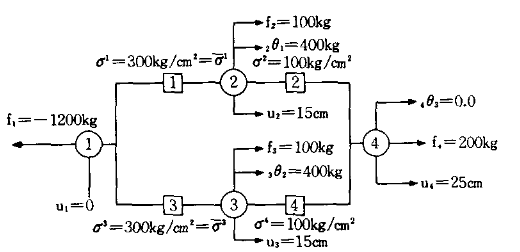
$Z={}_2\theta_1+{}_3\theta_2=200+0=200$kg

Fig. 7 (a) Results in Example 1



$Z={}_2\theta_1+{}_3\theta_2=0+450=450$kg

Fig. 7 (b) Results in Example 2



$Z={}_2\theta_1+{}_3\theta_2+{}_4\theta_3=400+400+0=800$kg

Fig. 7 (c) Results in Example 3

5. Conclusion

The FE (Finite Element) & LP (Linear Programming) method in optimal structural systems was described. Some numerical examples in assemblies of one-dimensional elastic bars are also presented. The tractability in both the boundary conditions and the equality or inequality constraints makes sure that the method becomes one of the most powerful techniques for several new types of boundary value problems in optimal structural design.

In order to save computer time and memory, an efficient computational algorithm of the FE&LP method is developed in the optimal structural systems. The details of the algorithm will be presented in another place.

Table 2(a) Results on Nodes

Example	Node Number	Displacement u (cm)	Controllable Load ${}_j\theta_i$ (kg)	${}_j\theta_i + f_n$ (kg)	Reaction Load (kg)
Example 1	①	0.0			$f_1 = -400.0$
	②	1.0	200.0	300.0	
	③	2.0	0.0	100.0	
	Total		$Z = 200.0$	400.0	
Example 2	①	0.0			$f_1 = -850.0$
	②	4.0	0.0	200.0	
	③	4.5	450.0	650.0	
	Total		$Z = 450.0$	850.0	
Example 3	①	0.0			$f_1 = -1200.0$
	②	15.0	400.0	500.0	
	③	15.0	400.0	500.0	
	④	25.0	0.0	200.0	
	Total		$Z = 800.0$	1200.0	

Table 2(b) Results on Elements

Example	Element Number	Stress σ^e (kg/cm$_2$)
Example 1	[1]	200.0($=\bar{\sigma}^1$)
	[2]	100.0
Example 2	[1]	400.0($=\bar{\sigma}^1$)
	[2]	100.0
	[3]	450.0
Example 3	[1]	300.0($=\bar{\sigma}^1$)
	[2]	100.0
	[3]	300.0($=\bar{\sigma}^3$)
	[4]	100.0

References

1) Futagami, T., "Development of Finite Element & Linear Programming Method and Its Application to Problems of Water Pollution Control," Proceedings, 19th Hydraulics Conference, Japan Society of Civil Engineers, February, 1975, pp. 133-138, (in Japanese).

2) Futagami, T., Tamai, N., and Yatsuzuka, M., "FEM Coupled with LP for Water Pollution Control," Journal of Hydraulics Division, ASCE, 102, HY7, 1976, pp. 881-897.

3) Futagami, T., "Several Mathematical Methods in Water Pollution Control—The Finite Element & Linear Programming Method and the Related Methods, "Thesis Presented to Kyoto University, in Partial Fullfillment of the Requirements for the Degree of Doctor of Phylosophy, February, 1976.

4) Futagami, T., Fukuhara, T., and Tomita, M., "Transient Finite Element & Linear Programming Method in Environmental Systems Control—The Efficient Computational Algorithm, Preprints, IFAC Environmental Systems Symposium, Kyoto, August, 1977, Pergamon Press, pp. 143-150.

5) Zienkiewicz, O. C., The Finite Element Method, third ed., McGraw-Hill, 1977.

6) Strang, G., and Fix, G., "An Analysis of Finite Element Method," Prentice Hall, 1973.

7) Finlayson, B. A., "The Method of Weighted Residuals and Variational Principles," Academic Press, 1972.

8) Dantig, B. G., "Linear Programming and Extensions," Princeton University Press, 1963.

9) Gass, I. S., "Linear Programming, Methods and Applications," 3rd ed., McGraw-Hill, Kogakusha, Ltd., 1969.

10) Bellman, R., "Dynamic Programming," Princeton University Press, 1957.

11) Futagami, T., "Dynamic Programming for a Sewage Treatment System," Proceedings, 5th International Water Pollution Research Conference, Pergamon Press, Spring 1971, pp. II-21/1-II-21/12.

12) Pontryagin, L. S., Boltyanskii, R. V., Gamkrelidze, R. V. and Mischenko, E. F., "The Mathematical Theory of Optimal Processes," Wiley-Interscience, 1962, (English Translation by Trirogoff).

13) Fan, L. T., "The Continuous Maximum Principle," John Wiley & Sons, 1966.

14) Aguado, E., and Remson, I., "Ground-Water Hydraulics in Aquifer Management," Journal of the Hydraulic Division, ASCE, Vol. 100, No. HY1, January, 1974, pp. 103-108.

15) Gallagher, R. H. and Zienkiewicz, O. C., "Optimal Structural Design," John Wiley & Sons, 1973.

16) Gutkowski, W., "Oprimization of Engineering Structures-Theory and Applications," Proceedings of 9th IFIP Conference on Optimization Techniques, 1979, pp. 14-29.