Adaptive Robust Control Schemes with a Simple Structure for Uncertain Time-Delay Systems with Unknown Saturated Input Nonlinearities

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Abstract: In this paper, the problem of robust stabilization is considered for a class of uncertain dynamical systems with completely unknown saturated inputs and nonlinear time-varying delayed state perturbations. The upper bounds of nonlinear the perturbations are assumed to be completely unknown nonlinear functions, and the time-varying delays are assumed to be any continuous and bounded nonnegative functions. It is also assumed that one need not know any information on the saturated input nonlinearities. Based on Wu inequality, a new design method is presented, whereby a class of adaptive robust state feedback control schemes with a relatively simple structure can be constructed. That is, such control laws do not involve completely the nonlinear upper bounds and the parameters of the saturation characteristics, and only consist of a conventional linear control law and a self-tuning control gain. Finally, an illustrative example is provided to demonstrate the validity of the presented design method.

Key Words: Uncertain systems, input saturation constraints, time-varying delays, adaptive control, nonlinear perturbations, exponential ultimate boundedness.

1. Introduction

In engineering control problems, it can not be avoidable to involve time delays and saturated input nonlinearities in the dynamical systems which describe some practical control plants. Therefore, the control problems have been widely considered for dynamical systems with time delays and saturated input nonlinearities (see, e.g. e.g. [1, 2, 3, 4, 5, 6, 16, 17] and the references therein).

For uncertain dynamical systems with time-delay, the problems of adaptive robust control have been considered, and some types of adaptive robust control laws have been also developed (see, e.g. [7, 8, 9, 10, 11, 12] and the references therein). For instance, in [7] an adaptive robust control law is proposed for uncertain systems with multiple time delays, and the uniform ultimate boundedness of the considered time-delay systems can be obtained. Furthermore, in [8] by introducing a so-called improved adaptation law with $\sigma$-modification, the proposed adaptive robust control law can also guarantee a asymptotical result for uncertain time-delay systems. In addition, [9], for uncertain nonlinear time-delay systems where the upper bounds of the delayed state perturbations are assumed to be bounded in some smooth $K$-class functions, an adaptive robust control law is also presented. However, it should also be pointed out that in the aforementioned references, the time delays have been assumed to be some nonnegative constants.
On the other hand, for uncertain dynamical systems with saturated input nonlinearities, an extensive research has been implemented, and some design methods have been proposed in the control literature. In [13], for example, a nonlinear feedback control method is developed for the linear dynamical systems. In [14], the method developed in [13] has been extended as a composite nonlinear feedback control law scheme with linear and nonlinear feedback laws which do not involve any switching elements. In [15], a design method with positive $\mu$-modification is also proposed. However, in our opinion, it seems that the design methods proposed in the aforementioned control literature are not direct, and the resulting control schemes are rather complicated. In some recent papers [16,17], a simple design method, called an adaptive design method, is developed to synthesize some control schemes for dynamical systems with input saturations. Moreover, in [17], the adaptive design method proposed in [16] is also applied to robust output tracking control problem of uncertain strict-feedback nonlinear systems with any input saturation, and some simple adaptive robust output tracking control schemes have been constructed.

In this paper, we consider the problem of robust stabilization for a class of uncertain dynamical systems with any saturated inputs and nonlinear time-varying delayed state perturbations. Here, we assume that the upper bounds of the perturbations are completely unknown and that the derivatives of time-varying delays are not require to have to be less than one. We also assume that it is not necessary to know any information on the saturated input nonlinearities. For such a class of uncertain dynamical systems, on the basis of Wu inequality proposed in [18], we present a novel method whereby a class of adaptive robust control laws with a relatively simple structure can be formed. That is, such control laws do not completely involve the nonlinear upper bounds and the parameters of the saturation characteristics, and only consist of a conventional linear control law and a self-tuning control gain. In particular, such closed-loop systems are globally stable in the presence of saturated inputs and nonlinear time-varying delayed state perturbations.

The rest of the paper is given as follows. After an introduction in this section, the problem formulation is described in Section 2. A novel design method is presented in Section 3 to deal with any saturated input nonlinearity and time-varying delays. A numerical example is given in Section 4, and the corresponding simulations are provided to demonstrate the validity of the theoretical results obtained in this paper. Finally, Section 5 will conclude the paper with some remarks.

2. Problem Formulation

Consider a class of uncertain dynamical systems with any saturation input nonlinearity and any nonlinear delayed state perturbations, described by

$$\frac{dx(t)}{dt} = A(t)x(t) + B(t) \left[ S_1(u(t)) + \sum_{j=1}^{p} \Delta g_j(x(t-\tau_j(t))) \right] \tag{1}$$

where $t \in R^+$ is the “time”, $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the control vector, $A(t)$ and $B(t)$ are two matrices of appropriate dimensions, and for any $j \in \{1,2,\ldots,p\}$, $\Delta g_j(\cdot): R^n \rightarrow R^m$ represents the delayed state perturbations. In particular, the nonlinear function $S_1(u(t)): R^m \rightarrow R^m$ represents any saturated inputs defined by

$$S_1(u(t)) = \left[ S_1^1(u_1(t)) \ S_1^2(u_2(t)) \ \cdots \ S_1^m(u_m(t)) \right]^T \tag{2a}$$
where for any \( j \in \{1,2,\ldots,m\} \),

\[
\mathcal{S}_j(t) = \text{sat} \left[ b_j^M, b_j^m; u_j(t) \right] = \begin{cases} b_j^M, & \text{if } u_j(t) \geq b_j^M \\ u_j(t), & \text{if } b_j^m \leq u_j(t) \leq b_j^M \\ b_j^m, & \text{if } u_j(t) \leq b_j^m \end{cases}
\]

where \( b_j^M \) and \( b_j^m \) are any constants, and it is assumed that \( b_j^m < b_j^M \). In particular, in the paper we do not require for \( b_j^M \) and \( b_j^m \) to be known. This means that these parameters are never involved in the constructed control schemes.

Moreover, for each \( j \in \{1,2,\ldots,p\} \), the time delay \( \tau_j(t) \) is assumed to be any nonnegative continuous bounded function, i.e. satisfying

\[
0 \leq \bar{\tau}_j(t) \leq \bar{\tau}_j, \quad j \in \{1,2,\ldots,p\}
\]

where \( \bar{\tau}_j \) and \( \bar{\tau}_j \) are any nonnegative constants, and \( \bar{\tau}_j \) is not required generally to be zero. For mathematical completeness, the initial condition is given as follows.

\[
x(t) = \mathcal{X}(t), \quad t \in [t_0 - \bar{\tau}, t_0]
\]

where \( \bar{\tau} = \max \{ \bar{\tau}_j, j = 1,2,\ldots,p \} \) and \( \mathcal{X}(t) \) is a continuous function on \([t_0 - \bar{\tau}, t_0]\). For the uncertain time-delay systems described by (1) and (2), without loss of generality, we assume that the pair \( \{A(\cdot), B(\cdot)\} \) is uniformly completely controllable, which will result in such a fact that for any given positive definite matrix \( Q \in \mathbb{R}^{n \times n} \) and any given positive constant \( \hat{\eta} \), the Riccati equation of the form

\[
\frac{dP(t)}{dt} = -P(t)A(t) - A^T(t)P(t) + \hat{\eta}P(t)B(t)B^T(t)P(t) - Q
\]

will have a positive definite solution \( P(t) \in \mathbb{R}^{n \times n} \) with

\[
\hat{\alpha}_1 I \leq P(t) \leq \hat{\alpha}_2 I
\]

where \( \hat{\alpha}_1 \) and \( \hat{\alpha}_2 \) are two positive numbers.

In addition, it should be pointed out that if \( A \) and \( B \) are two constant matrices, the aforementioned differential Riccati equation can be reduced to a much simpler algebraic Riccati equation in the form of

\[
A^T P + PA - \hat{\eta}PPBB^T P = -Q
\]

Furthermore, we also assume that for any \( j \in \{1,2,\ldots,p\} \), the uncertain nonlinear function \( \Delta g_j(\cdot): \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^m \) is bounded in Euclidean norm. Here, without loss of generality, we can assume that there exist an uncertain nonlinear function \( \tilde{\rho}_j(\cdot): \mathbb{R}^n \rightarrow \mathbb{R}^{\bar{\theta}_j} \) and an unknown constant vector \( \theta^* \in \mathbb{R}^{\bar{\theta}_j} \) such that

\[
\left\| \Delta g_j(x(t-\tau_j(t))) \right\| \leq \left( \theta^*_j \right)^T \tilde{\rho}_j(x(t-\tau_j(t)))
\]

where
\[
\begin{align*}
\beta_j(\cdot) &:= \left[ \beta_{j1}(\cdot) \beta_{j2}(\cdot) \cdots \beta_{jn}(\cdot) \right]^T \\
\theta^*_i &:= \left[ \theta^*_1 \theta^*_2 \cdots \theta^*_n \right]^T
\end{align*}
\]

and where for any \( j \in \{1,2,...,p\} \), \( \beta_j(x, t) > 0 \), \( i = 1,2,...,l_j \), for all \( x \) such that \( \|x\| > 0 \), and without loss of generality, \( \beta_j(\cdot) > 0 \), \( i = 1,2,...,l_j \), are assumed to be continuous, uniformly bounded with respect to time, and locally uniformly bounded with respect to the state \( x \).

Now, our main objective is to synthesize a class of simple state feedback control schemes that can guarantee the stability of uncertain time-delay systems described by (1) and (2) in the presence of any saturated inputs and any nonlinear delayed state perturbations.

**Remark 2.1.** The main purpose of this paper is to develop some simple and direct methods to deal with the time-varying delays and saturated input nonlinearities. For convenience, we use the dynamical systems with linear part, given in (1), with any time-varying delays and any saturated input nonlinearities defined in (2). It is obvious that the design method to be presented in this paper can be applied to other relatively complicated dynamical systems with such time-varying delays and saturated input nonlinearities.

In the paper, we will utilize a new integral inequality, called Wu inequality [18], to complete our stability analysis of the resulting adaptive closed-loop time-delay systems in the next section. Here, such an integral inequality is described by the following lemma.

**Lemma 2.1.** [18] Let \( \mathcal{Y}(t) \) and \( \mathcal{Z}(t) \) be any continuous functions with \( \mathcal{Y}(t_0) \neq 0 \), and let \( h_j(t) \) be any nonnegative function with \( 0 \leq h_j(t) \leq \bar{h}_j \) where \( \bar{h}_j \) is any positive constant. Moreover, \( \theta, \gamma, \beta_1, \beta_2, \) and \( \beta_3 \) are any given positive constants, and \( g_1(\mathcal{Y}, \mathcal{Z}) \) and \( g_2(\mathcal{Y}, \mathcal{Z}) \) are any nonnegative continuous functions. Then, there are always some positive constants \( \varepsilon_j, j = 1,2,...,n \), such that the inequality
\[
|\mathcal{Y}(t)| \leq \gamma e^{-\theta(t-t_0)} + \beta_1 \sum_{j=1}^{n} \varepsilon_j \int_{t_0}^{t} e^{-\theta(s-t)} g_1(\mathcal{Y}(s), \mathcal{Z}(s)) ds + \beta_2 \int_{t_0}^{t} e^{-\theta(s-t)} g_2(\mathcal{Y}(s-h_j(s)), \mathcal{Z}(s-h_j(s))) ds + \beta_3 \varepsilon_j
\]
implies that
\[
|\mathcal{Y}(t)| \leq \frac{\gamma}{1-\eta^*} e^{-\theta_0(t-t_0)} + \frac{\beta_3}{1-\eta^*}
\]
where \( \theta_0 \) and \( \eta^* \) are some positive constants which satisfy \( \theta_0 < \theta \) and \( \eta^* < 1 \), respectively.

### 3. Stabilizing Control Schemes

In this section, we utilize Wu inequality to develop a class of stabilizing control schemes with a relatively simple structure. Here, we propose the state feedback controller described by
\[
u(t) = \hat{\pi}(x(t), \hat{\kappa}(t), t) = -\frac{1}{2} \hat{\eta} \hat{\kappa}(t) \bar{B}^T(t) P(t) x(t)
\]
where \( \hat{\eta} \) is a given positive constant, and \( \hat{\kappa}(t) \) is a self-tuning control gain (or adaptive control gain) which is updated by the following adaptive law:
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where $\bar{\gamma}$ and $\sigma$ are any given positive constants, and the initial condition of $\hat{\kappa}(t)$ is given as $\hat{\kappa}(t_0) \geq 0$. In addition, $P(t) \in \mathbb{R}^{n \times n}$ is the solution of Riccati equation (5).

**Remark 3.1.** It is obvious that the feedback control scheme constructed by (11) with (12) is linear in the state and has a self-tuning control gain. Here, this self-tuning control gain has been employed to tune the information on the saturated input nonlinearities so that such a linear control law is able to guarantee the stability of uncertain dynamical systems with input saturation and time-varying delayed state perturbations. Actually, such an adaptive control schemes with a relatively simple structure can be well understood to the system designers, and can be implemented easily in practical engineering problems. Moreover, the proposed robust control law is memoryless, and the adaptive law given in (12) is completely independent of the time delays. Thus, we can know that the time-varying delays functions $\tau_j(t)$, $j = 1, 2, \ldots, p$, are not required to be known for the system designer, by our design method.

Furthermore, we can also rewrite (12) as the following error adaptation system:

$$
\frac{d\hat{\kappa}(t)}{dt} = -\bar{\gamma}\sigma\kappa(t) + \frac{1}{2}\bar{\gamma}\eta\left\|B^T(t)P(t)x(t)\right\|^2 - \bar{\gamma}\sigma\kappa^* 
$$

where $\hat{\kappa}(t) := \hat{\kappa}(t) - \kappa^*$, and $\kappa^*$ is an unknown positive constant which will be described later.

In the following, let $(x, \hat{\kappa})(t)$ be a solution of the closed-loop dynamical systems, we can have the following main results.

**Theorem 3.1.** Consider the uncertain dynamical systems with any saturation input nonlinearity and any nonlinear delayed state perturbations, described by (1) and (2). Then, the state feedback control schemes given in (11) with (12) can guarantee that the solutions of the considered systems are uniformly exponentially bounded. That is, the state $x(t)$ converges uniformly exponentially towards a ball.

**Proof:** Firstly, similar to [16] and [17], we rewrite the saturated input function $\mathcal{S}_j^*(u_j(t))$, $j \in \{1, 2, \ldots, m\}$, given in (2) as follows.

$$
\mathcal{S}_j^*(u_j(t)) = m_j^*(t)u_j(t) + l_j^*(t), \quad j = 1, 2, \ldots, m
$$

where $m_j^*(t)$ and $l_j^*(t)$ are respectively given as

$$
\begin{align*}
\frac{1}{|u_j(t)| + \varepsilon}, & \quad \text{if } u_j(t) \geq b_j^M \\
1, & \quad \text{if } b_j^M \leq u_j(t) \leq b_j^M \\
\frac{1}{|u_j(t)| + \varepsilon}, & \quad \text{if } u_j(t) \leq b_j^M
\end{align*}
$$

and

$$
\begin{align*}
\hat{f}_j(t), & \quad \text{if } u_j(t) \geq b_j^M \\
0, & \quad \text{if } b_j^M \leq u_j(t) \leq b_j^M \\
\hat{g}_j(t), & \quad \text{if } u_j(t) \leq b_j^M
\end{align*}
$$

where $\varepsilon$ is any positive constant, and $\hat{f}_j(t)$ and $\hat{g}_j(t)$ are two bounded functions satisfying that for any $t \in \mathbb{R}^+$. 

\[
\frac{u_j(t)}{|u_j(t)| + \varepsilon} + \dot{f}_j(t) = b_j^M
\]

and
\[
\frac{u_j(t)}{|u_j(t)| + \varepsilon} + \dot{g}_j(t) = b_j^\alpha
\]

Here, it is obvious that for any \( t \in R^+ \), \(|\dot{f}_j(t)| \leq |b_j^M| + 1 \), and \(|\dot{g}_j(t)| \leq |b_j^\alpha| + 1 \).

Thus, from (15)-(17), we can also introduce some positive parameters which are defined as follows.

That is, for any \( j \in \{ j = 1, 2, \ldots, m \} \),
\[
\eta_j^m := \min \left\{ m_{e_j}(t), \ t \in R^+ \right\} > 0
\]
\[
d_j^M := 1 + \max \left\{ |b_j^M|, |b_j^\alpha| \right\} > 0
\]

It follows that
\[
|l_j^\varepsilon(t)| \leq d_j^M, \quad j = 1, 2, \ldots, m
\]

Therefore, it is not difficult from (14)-(19) that the saturated input vector \( S_\gamma(u) : R^m \rightarrow R^m \), given in (2), can be redescribed as follows.
\[
S_\gamma(u(t)) = M^\varepsilon(t)u(t) + L^\varepsilon(t)
\]

where
\[
M^\varepsilon(t) := \text{diag}\left\{ m_1^\varepsilon(t), m_2^\varepsilon(t), \ldots, m_m^\varepsilon(t) \right\}
\]
\[
L^\varepsilon(t) := \begin{bmatrix} l_1^\varepsilon(t) & l_2^\varepsilon(t) & \cdots & l_m^\varepsilon(t) \end{bmatrix}^T
\]

It is obvious from (18) and (19) that for any \( t \geq t_0 \), the matrix \( M^\varepsilon(t) \) is positive definite, and the vector \( L^\varepsilon(t) \) is norm-bounded. Thus, the following definitions can be given.
\[
\xi^* := \min \left\{ \lambda_{\min}(M^\varepsilon(t)), \ t \in R^+ \right\} > 0
\]
\[
\zeta^* := \max \left\{ \lambda_{\max}
\left( L^\varepsilon(t)(L^\varepsilon(t))^T \right), \ t \in R^+ \right\} > 0
\]

Here, it is worth noting that there always are theoretically such positive constants \( \xi^* \) and \( \zeta^* \) and they are not required to be known.

Then, for the closed-loop time-delay systems with input saturation described by (1) and (11) with (12), a quasi-Lyapunov function is introduced as follows.
\[
V(x, \bar{\kappa}) = x^T(t)P(t)x(t) + \xi^* \bar{\kappa}^{-1}\kappa^2(t)
\]

By taking the derivative of \( V(\cdot) \) along the trajectories of the closed-loop time-delay systems with input saturation, we have that for any \( t \geq t_0 \),
\[
\frac{dV(x, \bar{\kappa})}{dt} = x^T(t)\left[ dP(t) \right] + A^T(t)P(t) + P(t)A(t)x(t)
\]
\[
+ 2x^T(t)P(t)B(t)S_\gamma(u(t)) + 2x^T(t)P(t)B(t)\sum_{j=1}^{p} \Delta g_j(x(t-\tau_j(t)))
\]
According to (5), (20) and (21), we can further obtain that for any \( t \geq t_0 \),

\[
\frac{dV(x, \hat{y})}{dt} = x^T(t) \left[ -Q + \tilde{\eta} P(t) B(t) B^T(t) P(t) \right] x(t) \\
+ 2x^T(t) P(t) B(t) \sum_{j=1}^{p} \Delta g_j(x(t-\tau_j(t))) \\
+ 2\xi^* \gamma^{-1} \kappa(t) \frac{d\kappa(t)}{dt}
\]

\[
\leq x^T(t) \left[ -Q + \tilde{\eta} P(t) B(t) B^T(t) P(t) \right] x(t) \\
+ 2x^T(t) P(t) B(t) M^e(t) u(t) \\
+ \zeta_0 \lambda_{\max} \left( L^e(t)(L^e(t))^T \right) \left\| B^T(t) P(t) x(t) \right\|^2 + \varsigma_0^{-1} \\
+ 2 \sum_{j=1}^{p} \left( \varsigma_j \left\| \tilde{\theta}_j \right\|^2 \left\| B^T(t) P(t) x(t) \right\|^2 + \varsigma_j^{-1} \left\| \hat{\rho}_j(x(t-\tau_j(t))) \right\|^2 \right) \\
+ 2\xi^* \gamma^{-1} \kappa(t) \frac{d\kappa(t)}{dt}
\]

\[
= -x^T(t) Q x(t) + 2x^T(t) P(t) B(t) M^e(t) u(t) \\
+ \tilde{\eta} \xi^* \left( \frac{1}{\eta^*} \left( \tilde{\eta} + \zeta_0 \xi^* + \sum_{j=1}^{p} \varsigma_j \left\| \tilde{\theta}_j \right\|^2 \right) \right) \left\| B^T(t) P(t) x(t) \right\|^2 \\
+ \sum_{j=1}^{p} \varsigma_j^{-1} \left\| \hat{\rho}_j(x(t-\tau_j(t))) \right\|^2 + \varsigma_0^{-1} \\
+ 2\xi^* \gamma^{-1} \kappa(t) \frac{d\kappa(t)}{dt}
\]

\[
= -x^T(t) Q x(t) + 2x^T(t) P(t) B(t) M^e(t) u(t) \\
+ \tilde{\eta} \xi^* \left( \frac{1}{\eta^*} \left( \tilde{\eta} + \zeta_0 \xi^* + \sum_{j=1}^{p} \varsigma_j \left\| \tilde{\theta}_j \right\|^2 \right) \right) \left\| B^T(t) P(t) x(t) \right\|^2 \\
+ \sum_{j=1}^{p} \varsigma_j^{-1} \left\| \hat{\rho}_j(x(t-\tau_j(t))) \right\|^2 + \varsigma_0^{-1} \\
+ 2\xi^* \gamma^{-1} \kappa(t) \frac{d\kappa(t)}{dt}
\]
\[ + \sum_{j=1}^{p} \zeta_j^{-1} \| \hat{\rho}_j(x(t-\tau_j(t))) \|^2 + \zeta_0^{-1} \]

\[ + 2\xi^*\gamma^{-1}\kappa(t) \frac{d\kappa(t)}{dt} \]

where

\[ \kappa^* := \frac{1}{\eta \xi^*} \left( \hat{\eta} + \zeta_0 \xi^* + \sum_{j=1}^{p} \zeta_j \| \theta_j^* \|^2 \right) \]  

Then, introducing (11), (13), and (21a) into (24) yields that for any \( t \geq t_0 \),

\[ \frac{dV(x, \kappa)}{dt} \leq -x^T(t)Qx(t) - \hat{\eta} \kappa(t)x^T(t)P(t)B(t)M^e(t)B^T(t)P(t)x(t) \]

\[ + \frac{\eta \xi^* \kappa^*}{B^T(t)P(t)x(t)} \]

\[ + \sum_{j=1}^{p} \zeta_j^{-1} \| \hat{\rho}_j(x(t-\tau_j(t))) \|^2 + \zeta_0^{-1} \]

\[ + 2\xi^*\gamma^{-1}\kappa(t) \frac{d\kappa(t)}{dt} \]

\[ \leq -x^T(t)Qx(t) - \hat{\eta} \kappa(t)\lambda_{\min}(M^e(t)) \left( B^T(t)P(t)x(t) \right)^2 \]

\[ + \frac{\eta \xi^* \kappa^*}{B^T(t)P(t)x(t)} \]

\[ + \sum_{j=1}^{p} \zeta_j^{-1} \| \hat{\rho}_j(x(t-\tau_j(t))) \|^2 + \zeta_0^{-1} \]

\[ + 2\xi^*\gamma^{-1}\kappa(t) \frac{d\kappa(t)}{dt} \]

\[ \leq -x^T(t)Qx(t) - \hat{\eta} \kappa(t) \left( B^T(t)P(t)x(t) \right)^2 \]

\[ + \frac{\eta \xi^* \kappa^*}{B^T(t)P(t)x(t)} \]

\[ + \sum_{j=1}^{p} \zeta_j^{-1} \| \hat{\rho}_j(x(t-\tau_j(t))) \|^2 + \zeta_0^{-1} \]

\[ + 2\xi^*\gamma^{-1}\kappa(t) \frac{d\kappa(t)}{dt} \]

\[ \leq -x^T(t)Qx(t) - \xi^* \sigma \kappa^2(t) \]

\[ + \sum_{j=1}^{p} \zeta_j^{-1} \| \hat{\rho}_j(x(t-\tau_j(t))) \|^2 + \zeta_0^{-1} + \xi^* \sigma \kappa^* \]

\[ = -x^T(t)Qx(t) - \xi^* \sigma \kappa^2(t) \]

\[ + \sum_{j=1}^{p} \zeta_j^{-1} \| \hat{\rho}_j(x(t-\tau_j(t))) \|^2 + \beta^* \]  

(26)

where \( \beta^* \) is an unknown positive parameter described by
Since $Q$ is a positive definite matrix, it is not difficult to obtain that for any $t \geq t_0$,
\[
\frac{dV(x, \tilde{\kappa})}{dt} \leq -\lambda_{\min}(Q)\hat{\alpha}_2^{-1}x^T(t)P(t)x(t) - \xi^*\sigma \hat{\kappa}^2(t) + \sum_{j=1}^{p} \zeta_j^{-1}\|\hat{\rho}_j(x(t-\tau_j(t)))\|^2 + \beta^*
\]
(28)

Then, letting
\[
\theta_{\min} := \min\{\lambda_{\min}(Q)\hat{\alpha}_2^{-1}, \sigma \hat{\gamma}\} > 0
\]
(29)
we can obtain from (22) and (29) that for any $t \geq t_0$,
\[
\frac{dV(t)}{dt} \leq -\theta_{\min}V(t) + \sum_{j=1}^{p} \zeta_j^{-1}\|\hat{\rho}_j(x(t-\tau_j(t)))\|^2 + \beta^*
\]
(30)

where $V(t) := V(x(t), \tilde{\kappa}(t))$. Thus, it follows from (6), (22), and (30) that for any $t \in \mathbb{R}^+$,
\[
\|x(t)\|^2 \leq \exp\{-\theta_{\min}(t-t_0)\} \hat{\alpha}_1^{-1}V(t_0) + \sum_{j=1}^{p} \hat{\alpha}_1^{-1}\zeta_j^{-1} \int_{t_0}^{t} \exp\{-\theta_{\min}(t-s)\} \|\hat{\rho}_j(x(s-\tau_j(s)))\|^2 ds + \beta^*\hat{\alpha}_1^{-1}\theta_{\min}^{-1}
\]
(31)

Therefore, by employing Wu inequality described in Lemma 2.1, it can be directly obtained from (31) that for any $t \geq t_0$,
\[
\|x(t)\|^2 \leq \hat{\alpha}_1^{-1}V(t_0) e^{-\eta_0(t-t_0)} + \frac{\beta^*\hat{\alpha}_1^{-1}\theta_{\min}^{-1}}{1 - \eta^*}
\]
(32)
where $\theta_0$ and $\eta^*$ are some positive constants which satisfy $\theta_0 < \theta_{\min}$ and $\eta^* < 1$, respectively.

Thus, it is obvious that the solutions $\|x(t)\|$ of the closed-loop time-delay systems converge exponentially to $B(\rho_0)$ where
\[
B(\rho_0) := \left\{ x : \|x\| \leq \rho_0 := \sqrt{\frac{\beta^*\hat{\alpha}_1^{-1}\theta_{\min}^{-1}}{1 - \eta^*}} \right\}
\]
That is, the solutions of the closed-loop time-delay systems are uniformly exponentially bounded. Thus, we complete the proof of Theorem 3.1.

Remark 3.2. In the proof of Theorem 3.1, many positive constants, such as $\xi^*, \zeta^*, \zeta_j, \kappa^*, \theta_{\min}, \eta^*$, and so on, have been used to obtain the bound of the states described by (32). However, it should be pointed out that such parameters have been never employed in the control scheme given in (11) with (12). In fact, such parameters are used only for the theoretical proof.

Remark 3.3. Though the parameters $b_1j$ and $b_2j$ of the saturation characteristics have been employed to define some positive constants, the presented control laws are completely independent of such positive constants. On the other hand, in the relative control literature, the time delay is often assumed either to be a
positive constant, or to be a nonnegative continuous function in time which needs that its derivative is less than one, i.e. \( \hat{\tau}_j(t) < 1, j = 1, 2, \ldots, p \). Under such an assumption, one often employ the Lyapunov-Krasovskii functional candidate to analyse the stability of the closed-loop time-delay systems. In this paper, this strict assumption is well relaxed. In fact, we only assume that the time-varying delay is any nonnegative continuous and bounded functions, and even its derivatives at some points may not exist.

**Remark 3.4.** In the light of (25), (27), (29), and (32), it can be observed that by adjusting the control parameters \( \hat{\eta}, \hat{\gamma}, \) and \( \sigma \), one can changes the radius \( \rho_0 \) of a ball \( B(\rho_0) \). Thus, in practical engineering control problems, we can use such control parameters to adjust and improve the performance of engineering control systems.

### 4. Numerical Example

In this section, we employ a numerical example to illustrate our synthesis method. Here, such a numerical example is given by the following uncertain time-delay systems with a saturated input.

\[
\frac{dx(t)}{dt} = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (\mathcal{S}_\gamma(u(t)) + \sum_{j=1}^{3} \Delta q_j(x(t-\tau_j(t)))) \tag{33a}
\]

where

\[
\mathcal{S}_\gamma(u(t)) = \text{sat}\left[b^M, b^m; u(t)\right]
\]

\[
= \begin{cases} 
  b^M, & \text{if } u(t) \geq b^M \\
  u(t), & \text{if } b^m \leq u(t) \leq b^M \\
  b^m, & \text{if } u(t) \leq b^m
\end{cases} \tag{33b}
\]

Here, letting that \( \hat{\eta} = 2.0 \) and \( Q = \text{diag}\{2, 2\} \), from the algebraic Riccati matrix equation, it can be obtained that

\[
P = \begin{bmatrix} 2.383 & 0.684 \\ 0.684 & 8.973 \end{bmatrix}
\]

The control parameters are given as follows.

\[
\sigma = 1.0, \quad \hat{\gamma} = 0.1
\]

Thus, for the dynamical systems with a saturated input, described by (36), from (5) with (6) we can obtain a continuous state feedback controller, by which the uniform ultimate boundedness of dynamical system (36) can be guaranteed.

For simulations, the time-varying delays \( \tau_j(t), j = 1, 2, 3 \), are depicted in Figure 1, where \( \tau_3(t) = 1 + 0.5 \sin(\pi t) \). Note that \( \tau_1(t), \tau_2(t), \) and \( \tau_3(t) \) are any continuous and bounded nonnegative functions, and the derivative of \( \tau_1(t) \) (or \( \tau_2(t) \)), i.e. \( \hat{\tau}_1(t) \) (or \( \hat{\tau}_2(t) \)), is not defined at \( t = 0.5n \) (or \( t = 0.25n \), \( n = 1, 2, \ldots \)).
Moreover, for simulation, the delayed state perturbations, the parameters of the saturation characteristics, and initial conditions are given as follows.

\[
\begin{align*}
\Delta g_1(x(t-\tau_1(t))) &= \sin(x_1(t-\tau_2(t)))e^{\tau_1(t)} \\
\Delta g_2(x(t-\tau_2(t))) &= \sin(x_2(t-\tau_2(t)))e^{\tau_2(t)} \\
\Delta g_3(x(t-\tau_3(t))) &= \sin(x_3(t-\tau_3(t)))e^{\tau_3(t)} \\
\end{align*}
\]

\[b^M = 3.0, \quad b^m = -5.5\]

\[x(0) = [-3.0, 2.0]^T, \quad \dot{x}(0) = 5.0\]

With the chosen parameter settings, the results of simulation are shown in Figure 2 — Figure 4 for this numerical example.
Figure 3. History of the control scheme $u(t)$ (dash) and the input saturation $S_\gamma(u(t))$ (solid).

Figure 4. History of the updating parameter $\hat{\kappa}(t)$.

The responses of the states of the systems are depicted in Figure 2 that shows that the systems described by (33) are stable, i.e. the states of (33) are uniformly exponentially bounded in the presence of time-varying delays and saturated input. In addition, the signals of the control input $u(t)$ and the input saturation function $S_\gamma(u(t))$ are also depicted shown in Figure 3. It is also shown from Figure 4 that the adaptation law with $\sigma$-modification makes the estimate values of the unknown parameters decreasing. In particular, it can known from Figure 2 that the closed-loop dynamical systems with time-varying delays and saturated input have a rather good dynamical performance.

5. Concluding Remarks

In this paper, we have discussed the problem of robust stabilization for a class of uncertain dynamical systems with any saturated inputs and nonlinear time-varying delayed state perturbations. We have assumed
that the upper bounds of the perturbations are completely unknown and that the derivatives of time-varying delays are not require to have to be less than one. We have also assumed that it is not necessary to know any information on the saturated input nonlinearities. The basis of Wu inequality, we have presented a novel method whereby a class of feedback control laws with a relatively simple structure can be formed. It has been shown that the formed control laws can guarantee a globally stable result in the presence of saturated inputs and nonlinear time-varying delayed state perturbations. Finally, a numerical examples has been also given and the corresponding simulations are implemented. It is shown from this numerical example and the results of its simulation that the results obtained in the paper are effective and feasible.

In particular, by combining the presented design method with other control methods, e.g. sliding mode control, switched control, fuzzy control, $H_{\infty}$ control, neural networks approximation, and so on, we may expect to obtain a number of interesting results for a rather large class of dynamical systems with time-varying delays and saturation in the actuator, which will be our further research works in the future.

References

[12] “Adaptive robust stabilisation for a class of uncertain nonlinear time-delay dynamical systems,”


